

# Gaussian Process Models for Nonlinear Time Series

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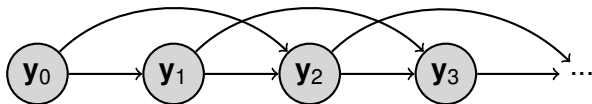
# Outline

- ▶ Time series models.
- ▶ Bayesian & nonparametric & nonlinear.
- ▶ A zoo of GP-based models.
- ▶ Predictions in GP state-space models.
- ▶ ...



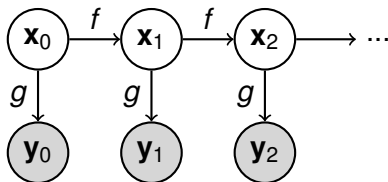
# Classic Models of Time Series

Auto-regressive model (AR, ARX, NARX...)



$$\mathbf{y}_t = f(\mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-n_y}) + \delta_t.$$

State-space models (SSM)



$$\mathbf{x}_{t+1} = f(\mathbf{x}_t) + \epsilon_t,$$

$$\mathbf{y}_t = g(\mathbf{x}_t) + \nu_t.$$

# Bayesian

Model uncertainty.

Controlled overfitting.

No need to artificially limit the complexity of the models. There is no statistical price to pay for adding more parameters.

# Nonparametric

Flexible.

Data is not condensed into a finite set of parameters.

# Nonlinear

The world is nonlinear!

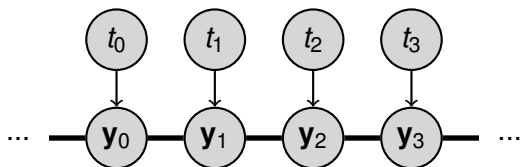
Linear dynamical systems are boring.

# A Zoo of GP-Based Time Series Models

- ▶ Linear-Gaussian Auto-Regressive / State-Space
- ▶ Nonlinear Auto-Regressive Model with GP
- ▶ State-Space Model with Transition GP
- ▶ State-Space Model with Transition and Emission GPs
- ▶ GP-LVM with Correlated Latent Variables

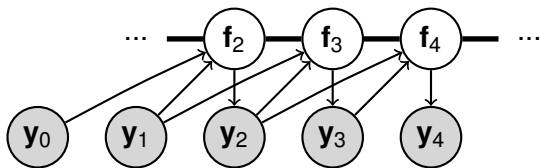


# 0. Linear-Gaussian Auto-Regressive / State-Space



$$\mathbf{y}(t) \sim \mathcal{GP}(m(t), k(t, t')).$$

# 1. Nonlinear Auto-Regressive Model with GP



$$f(\mathbf{Y}) \sim \mathcal{GP}(m_f(\mathbf{Y}), k_f(\mathbf{Y}, \mathbf{Y}')),$$

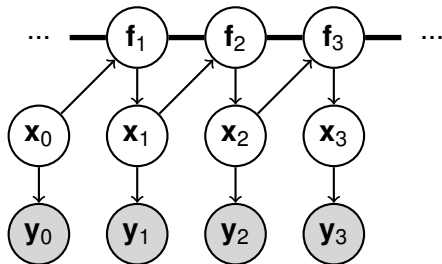
$$\mathbf{f}_t = f(\mathbf{Y}_{t-1}),$$

$$\mathbf{y}_t | \mathbf{f}_t \sim p(\mathbf{y}_t | \mathbf{f}_t, \boldsymbol{\theta}),$$

where

$$\mathbf{Y}_{t-1} = \{\mathbf{y}_{t-1}, \dots, \mathbf{y}_{t-n_y}\}.$$

## 2. State-Space Model with Transition GP (GP-SSM)



$$f(\mathbf{x}) \sim \mathcal{GP}(m_f(\mathbf{x}), k_f(\mathbf{x}, \mathbf{x}')),$$

$$\mathbf{x}_0 \sim p(\mathbf{x}_0),$$

$$\mathbf{f}_t = f(\mathbf{x}_{t-1}),$$

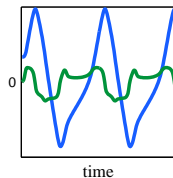
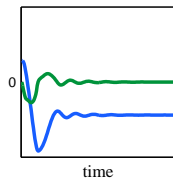
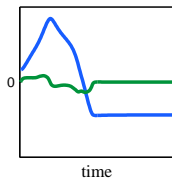
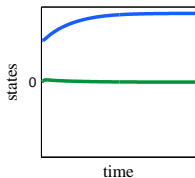
$$\mathbf{x}_t | \mathbf{f}_t \sim \mathcal{N}(\mathbf{x}_t | \mathbf{f}_t, \mathbf{Q}),$$

$$\mathbf{y}_t | \mathbf{x}_t \sim p(\mathbf{y}_t | \mathbf{x}_t, \theta_y).$$

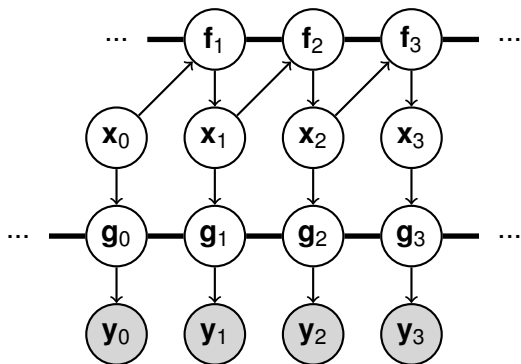
## 2. State-Space Model with Transition GP (GP-SSM)

### (GP-SSM)

4 independent state trajectories from a 2-state GP-SSM with *fixed* hyperparameters.



### 3. State-Space Model with Transition and Emission GPs (GP-SSM)



$$f(\mathbf{x}) \sim \mathcal{GP}(m_f(\mathbf{x}), k_f(\mathbf{x}, \mathbf{x}')), \quad g(\mathbf{x}) \sim \mathcal{GP}(m_g(\mathbf{x}), k_g(\mathbf{x}, \mathbf{x}')),$$

$$\mathbf{x}_0 \sim p(\mathbf{x}_0),$$

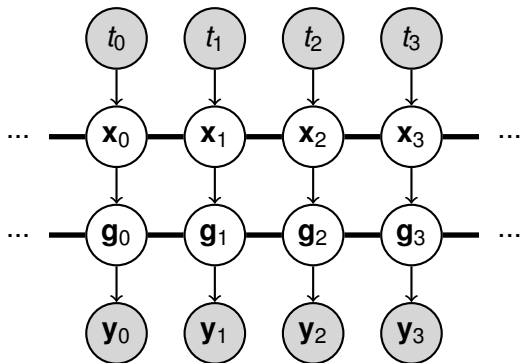
$$\mathbf{g}_t = g(\mathbf{x}_t),$$

$$\mathbf{f}_t = f(\mathbf{x}_{t-1}),$$

$$\mathbf{y}_t | \mathbf{g}_t \sim \mathcal{N}(\mathbf{y}_t | \mathbf{g}_t, \mathbf{R}).$$

$$\mathbf{x}_t | \mathbf{f}_t \sim \mathcal{N}(\mathbf{x}_t | \mathbf{f}_t, \mathbf{Q}),$$

## 4. GP-LVM with Correlated Latent Variables



$$\mathbf{x}(t) \sim \mathcal{GP}(m_{\mathbf{x}}(t), k_{\mathbf{x}}(t, t')),$$

$$\mathbf{g}(\mathbf{x}) \sim \mathcal{GP}(m_{\mathbf{g}}(\mathbf{x}), k_{\mathbf{g}}(\mathbf{x}, \mathbf{x}')),$$

$$\mathbf{x}_t = \mathbf{x}(t),$$

$$\mathbf{g}_t = \mathbf{g}(\mathbf{x}_t),$$

$$\mathbf{y}_t | \mathbf{g}_t \sim \mathcal{N}(\mathbf{y}_t | \mathbf{g}_t, \beta^{-1} \mathbf{I}).$$