Bayesian Nonparametric Nonlinear System Identification

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Bayesian System Identification. Why?

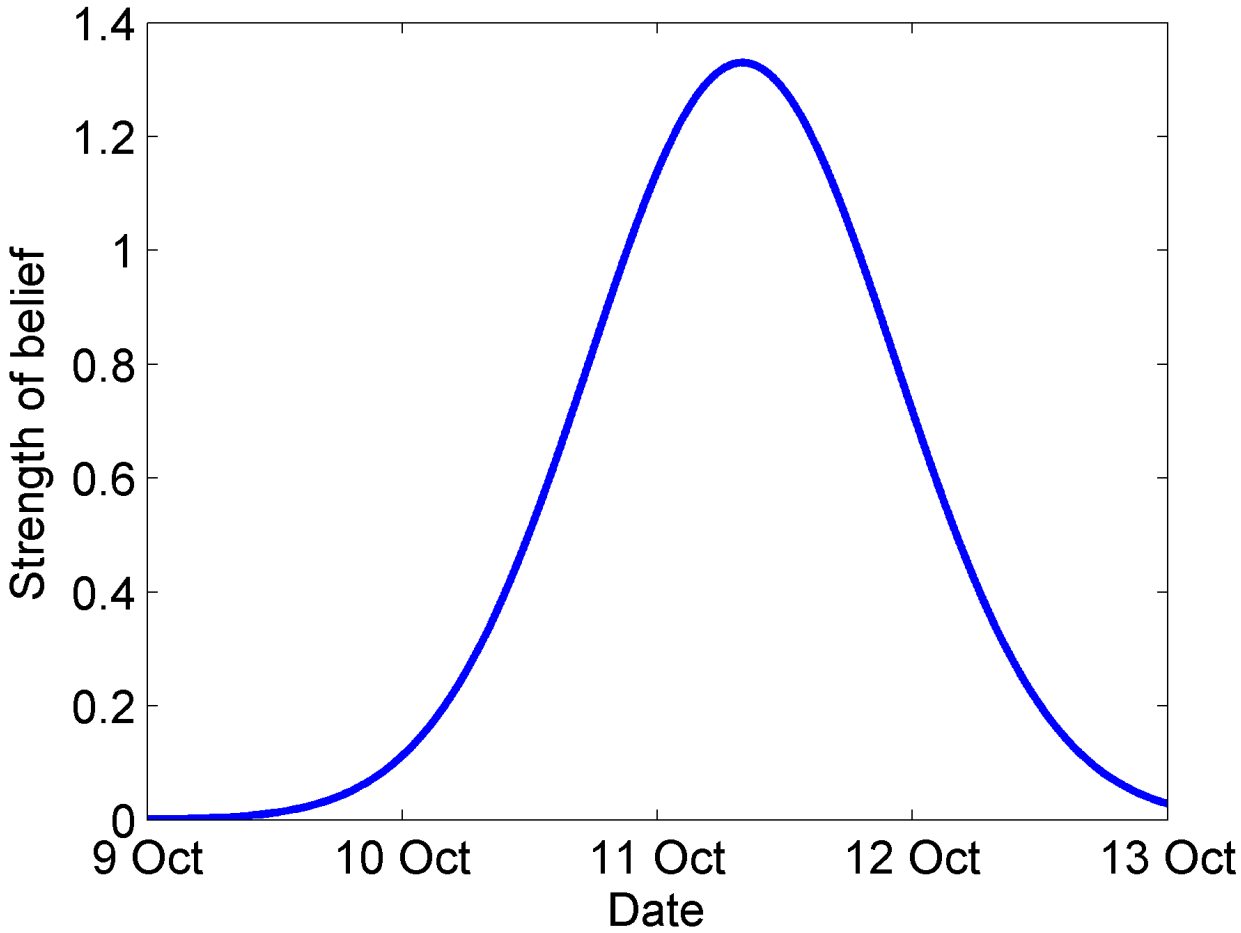
Introduction to Bayesian Modelling and Inference

Uses probability to *quantify uncertainty*.

Related to information rather than repeated trials.

Uncertainty is subjective, it depends on what we have seen.

Subjective Uncertainty



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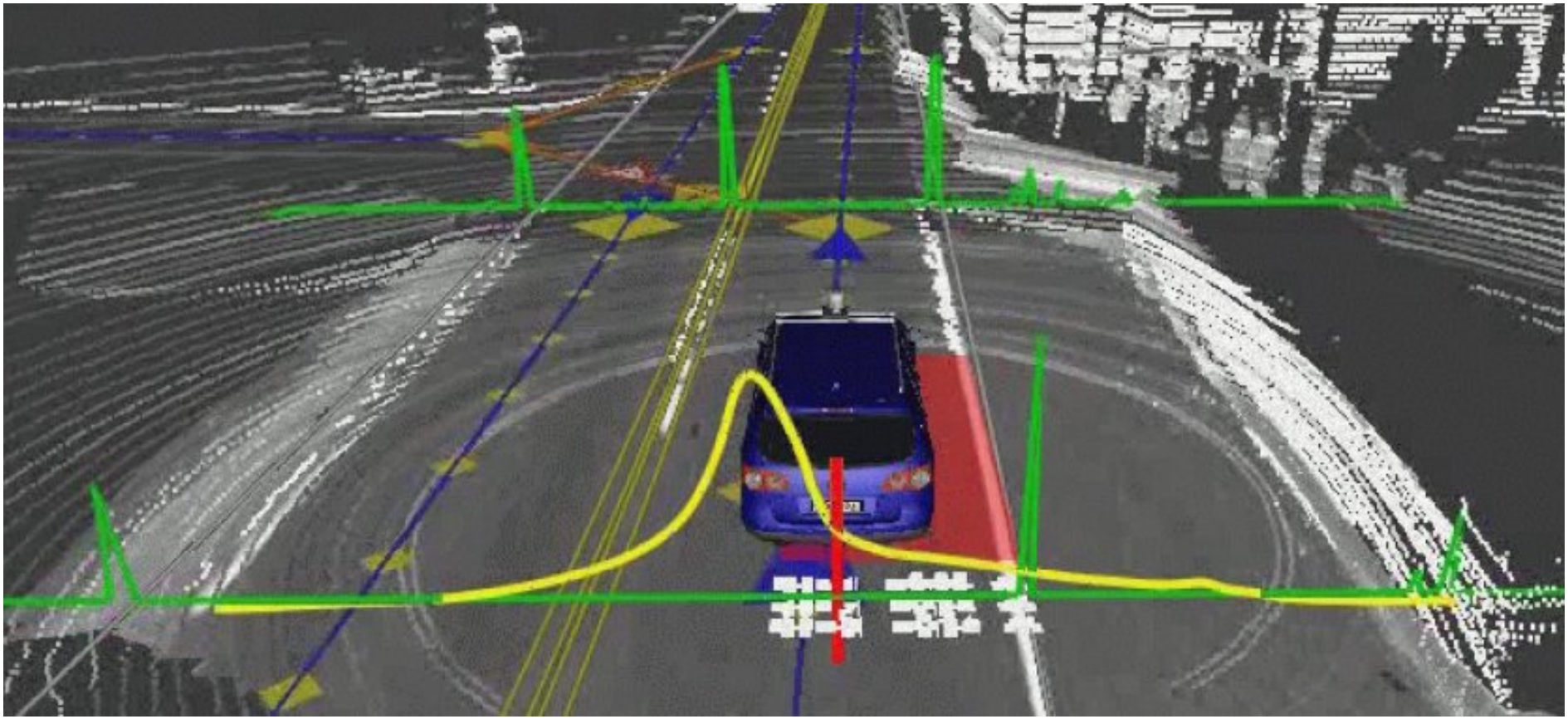

Stanford's self-driving car for the DARPA Urban Challenge (2007).

Bayesian Inference of Power and Drag

Infer power and drag based on noisy *acceleration measurements* using a simple inertial and aerodynamic model

$$a_t = \frac{1}{mV_t} P - \frac{\rho V_t^2 S}{2m} C_d$$

e.g. with Gaussian noise

$$y_t = \mathcal{N}(a_t, \sigma^2)$$

$$\mathcal{D} = \{y_1, \ldots, y_N, V_1, \ldots, V_N\}$$

Bayesian Inference of Power and Drag

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GOAL: find probability distribution of unknown parameters given data

$$p(\theta \mid \mathcal{D}) = \frac{p(\mathcal{D} \mid \theta)\ p(\theta)}{p(\mathcal{D})}$$

$$p(\theta \mid \mathcal{D}) \propto p(\mathcal{D} \mid \theta)\ p(\theta)$$

Bayesian Inference of Power and Drag

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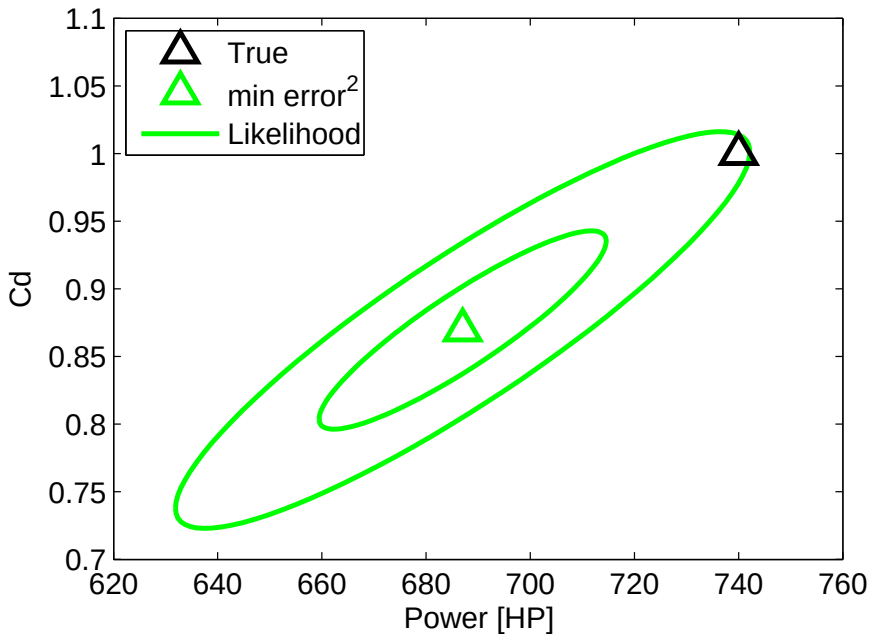

$$p(\mathcal{D} \mid \theta)$$

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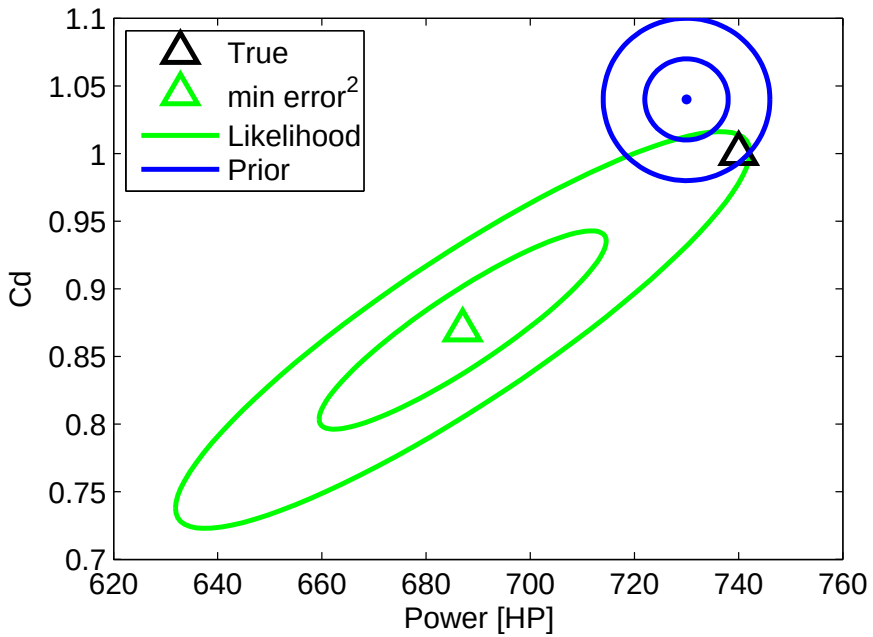

$p(\theta)$

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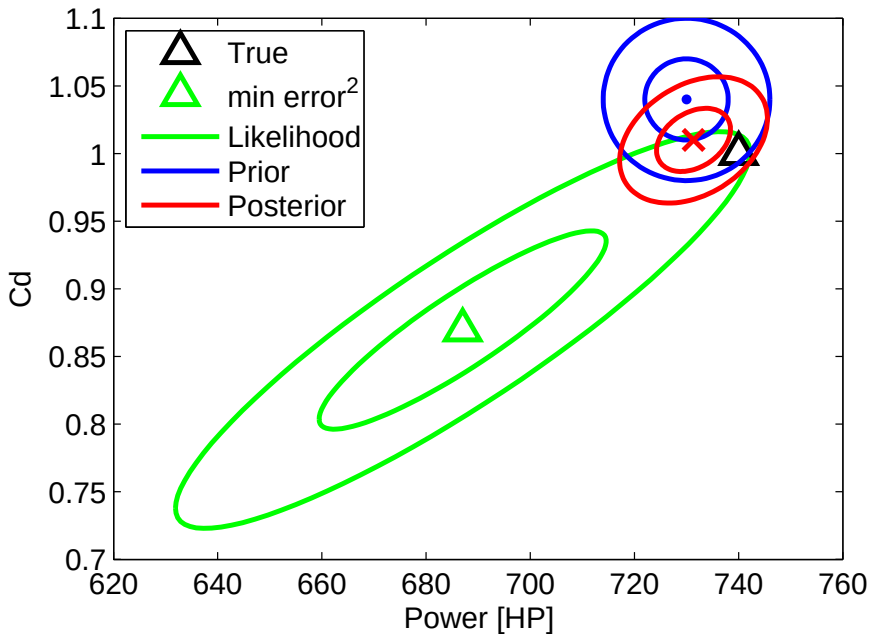


$$p(\theta \mid \mathcal{D})$$

Bayesian Inference

One could say that we have used the prior as a regulariser to solve

$$\theta^* = \arg\min_{\theta} \; L(\theta, \mathcal{D}) + J(\theta)$$

But, we were simply looking for the posterior: $p(\theta \mid \mathcal{D})$.

No optimisation!

The posterior represents our uncertainty and tells us how to average different models.

Bayesian Inference

What is the outcome of Bayesian inference?

Thomas' indoor localisation example.

Posterior over parameters $\rightarrow$ posterior over identified systems.

In fact, we can find posteriors over many different kinds of objects: functions, genetic trees, English language sentences, etc.

Bayesian Inference: Making Predictions

The posterior represents our uncertainty over the parameters.

Any prediction can be found by *averaging* over the posterior

$$p(\text{LapTime} \mid \mathcal{D}) = \int p(\text{LapTime}, \theta \mid \mathcal{D})\ d\theta$$
$$= \int p(\text{LapTime} \mid \theta)\ p(\theta \mid \mathcal{D})\ d\theta.$$

Making Decisions Under Uncertainty

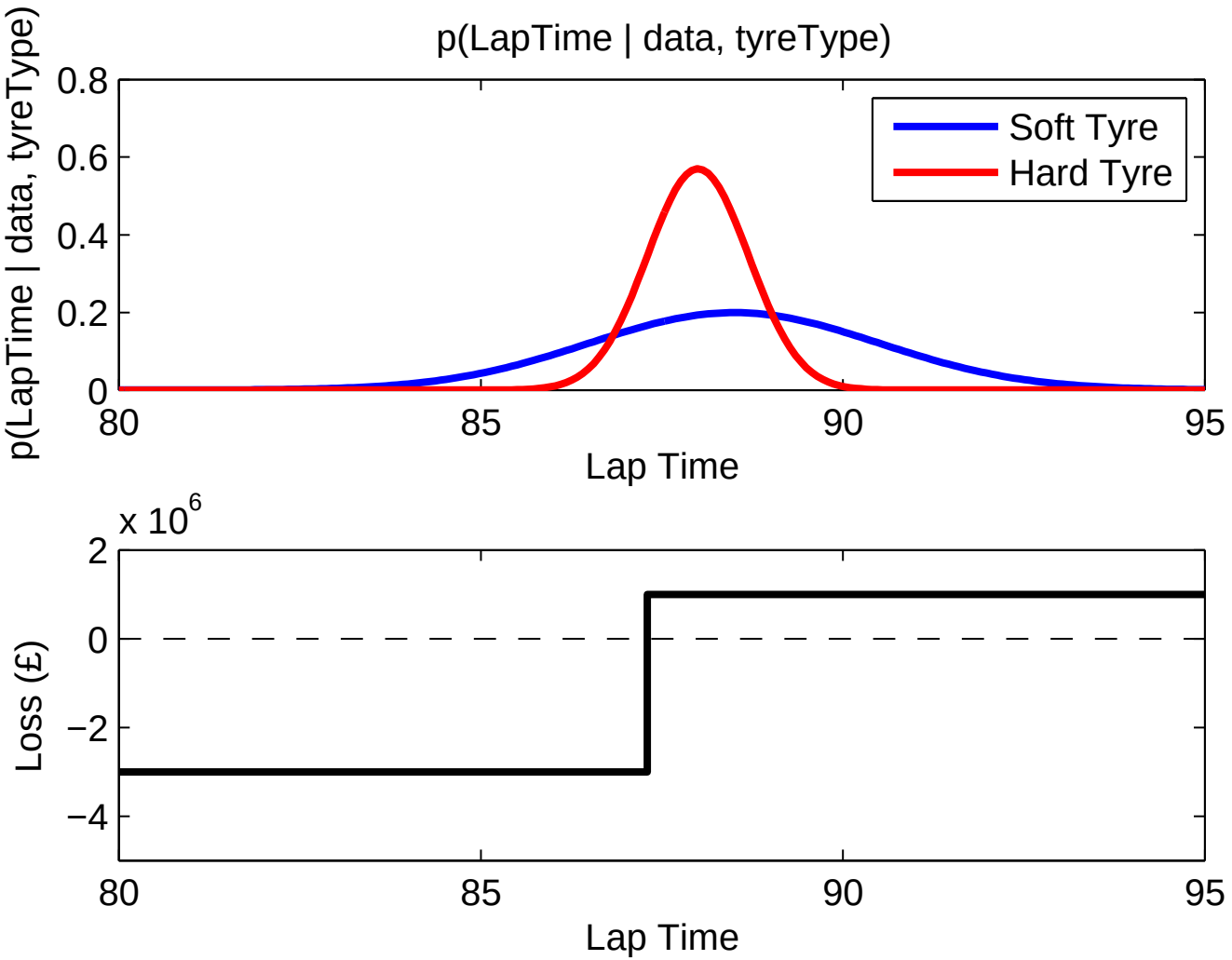
Bayesian inference provides probability distributions.

But, often, we can only take one action.

One solution: take the action that minimises the expected loss (aka risk) under the uncertainty provided by Bayesian inference.

$$a_{\text{opt}} = \arg\min_{a} \int \text{Loss}(a, \theta)\, p(\theta \mid \mathcal{D})\, d\theta$$

Making Decisions Under Uncertainty



Expected loss for hard tyre: $£3.65 \cdot 10^5$

Expected loss for soft tyre: $£-0.98 \cdot 10^5$

Over-fitting and Counting Parameters

Bayesian methods do not overfit because there is *no fitting!*

Inference is based on integration, i.e. averaging.

There is no statistical price to pay for adding more parameters.

Nonidentifiability is not a problem when making predictions.

From Formula 1 to Functions

$$y = f(x) \tag{1}$$

Task: Model f in a Bayesian manner

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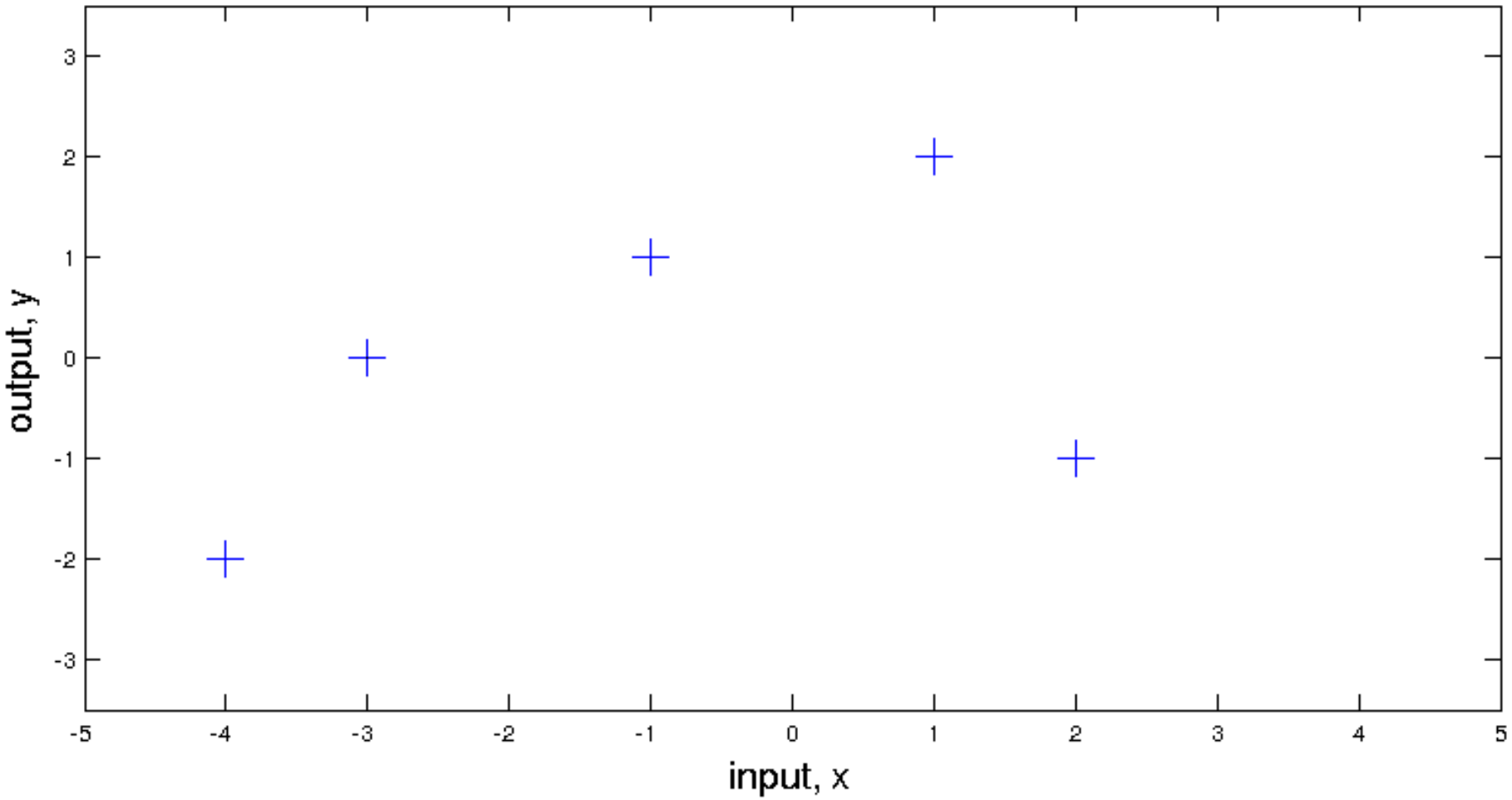

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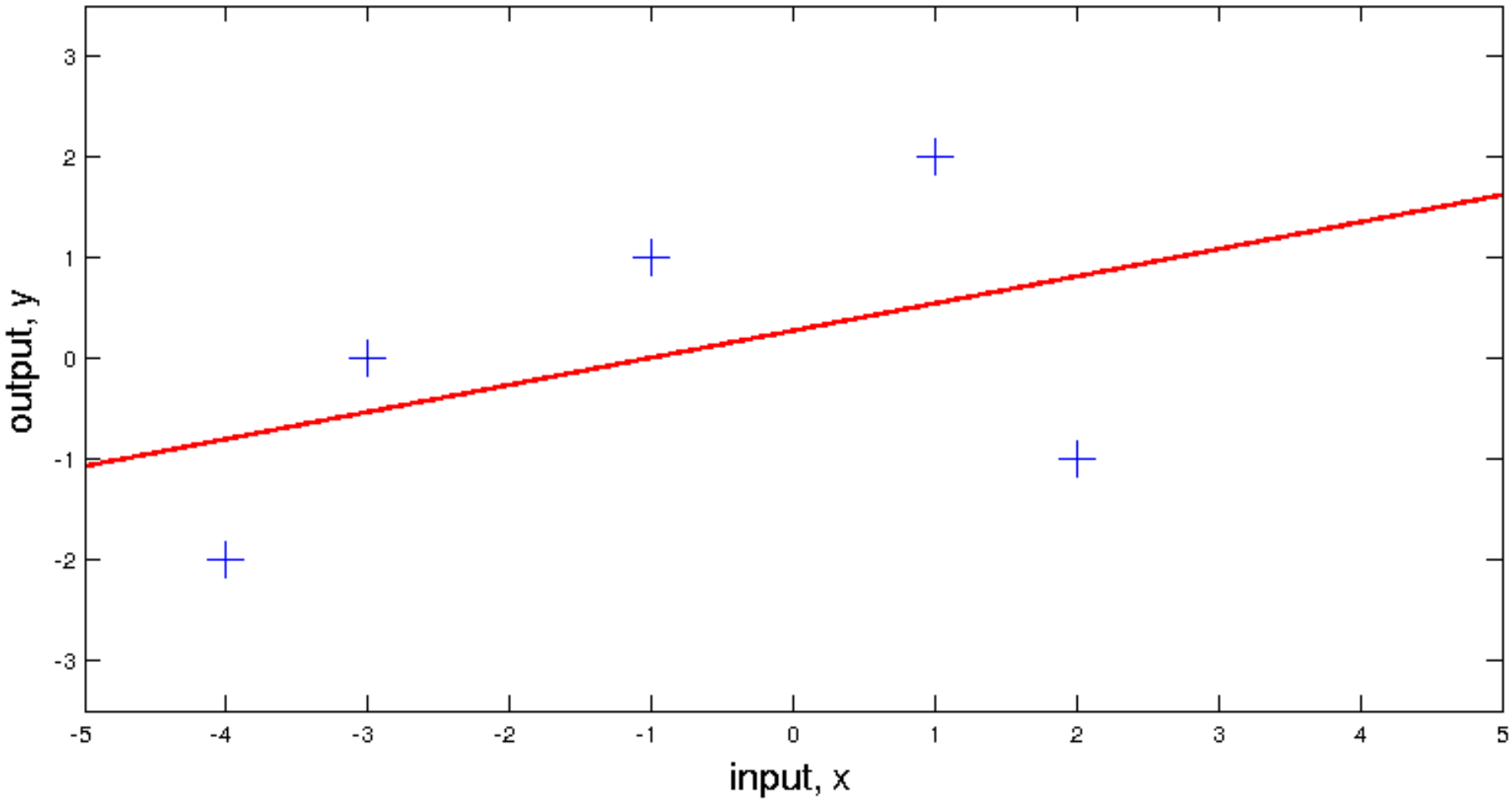

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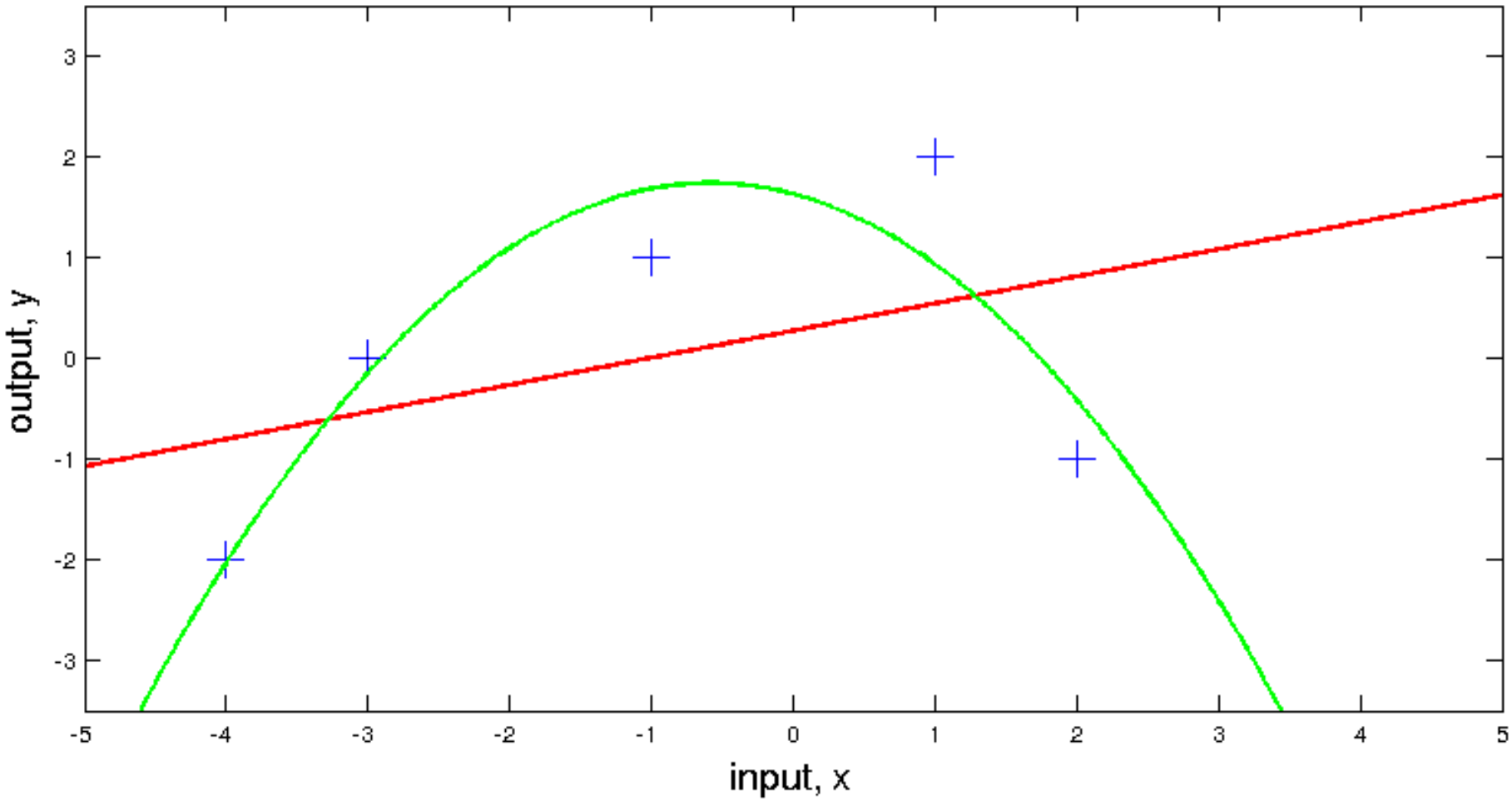

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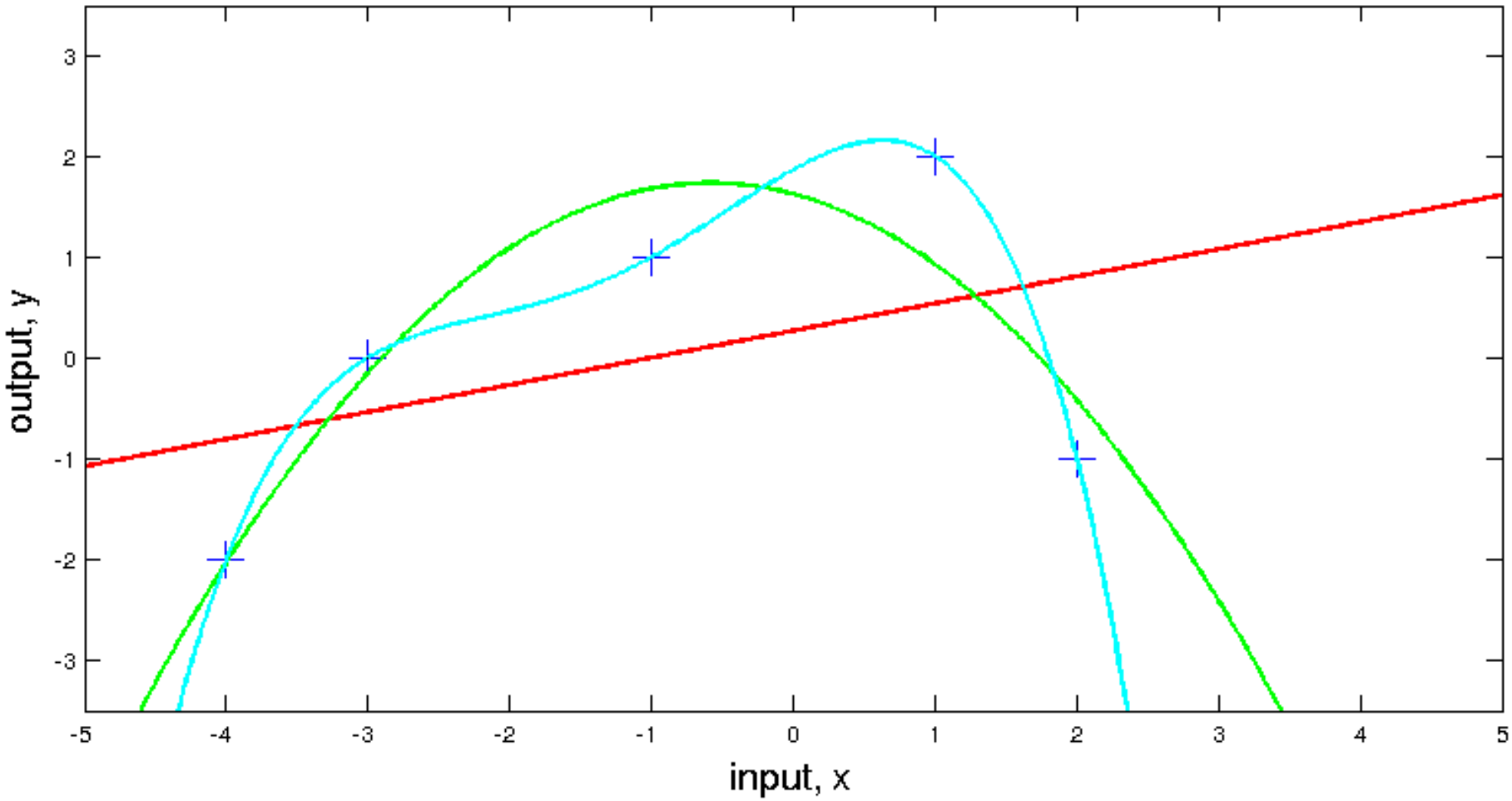

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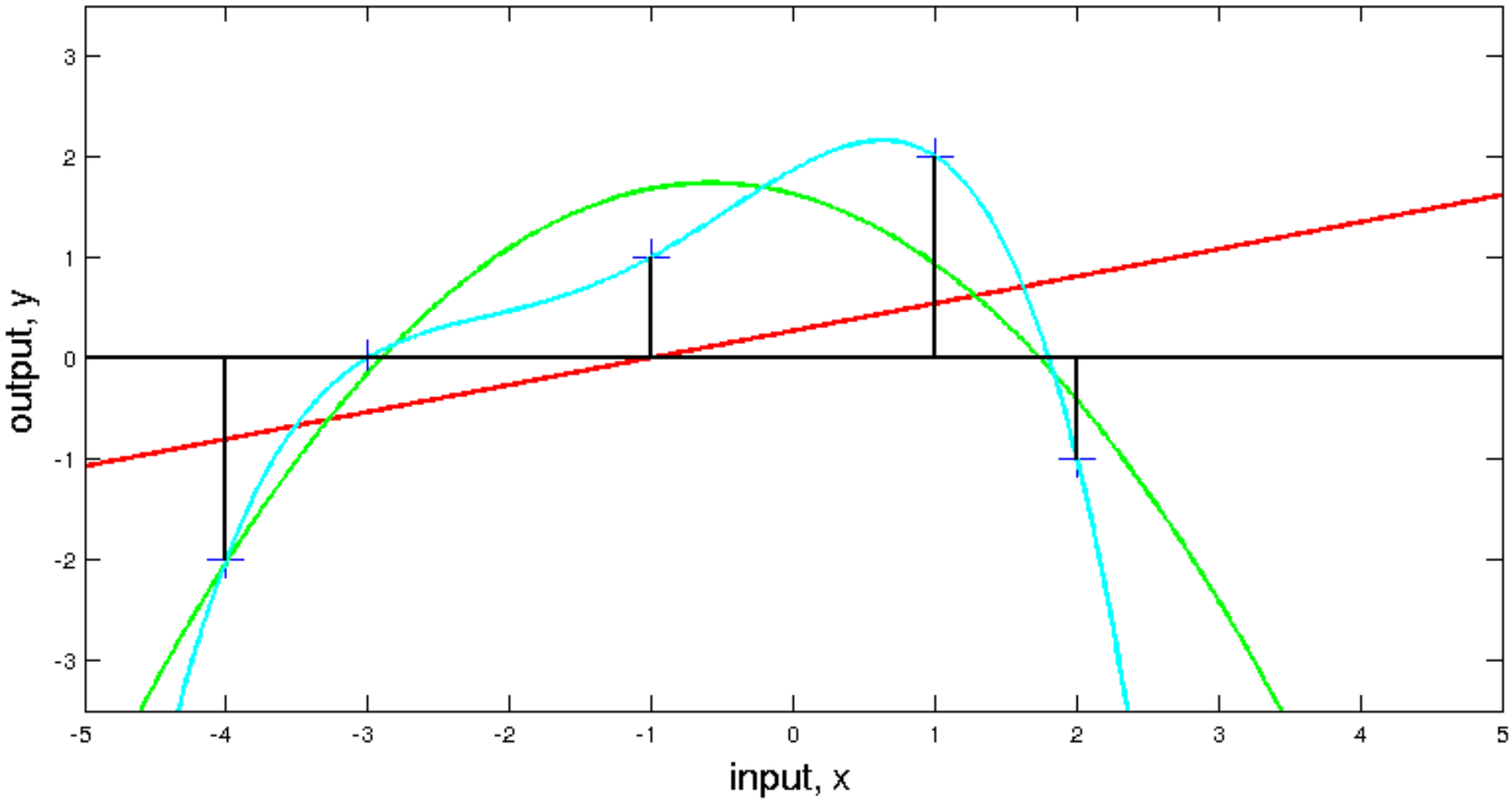

Nonlinear Models

Parametric nonlinear models,

- Polynomial, $f(x) = \sum_{i=0}^{N} w_i x^i$
- Radial Basis Function, $f(x) = \sum_{i=1}^{N} w_i \phi(x - c_i)$
- Fourier, $f(x) = \sum_{i=1}^{N} a_i \sin(w_i x - c_i)$

Problems,

- Either have to fix N or put a prior on it
- Can have a very large number of parameters to estimate
- Inference can be difficult
- Unclear how to choose priors

Nonparametric Models

Allow you to do inference on function values directly

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Parameterised directly by the training data

Nonparametric Models

Allow you to do inference on function values directly

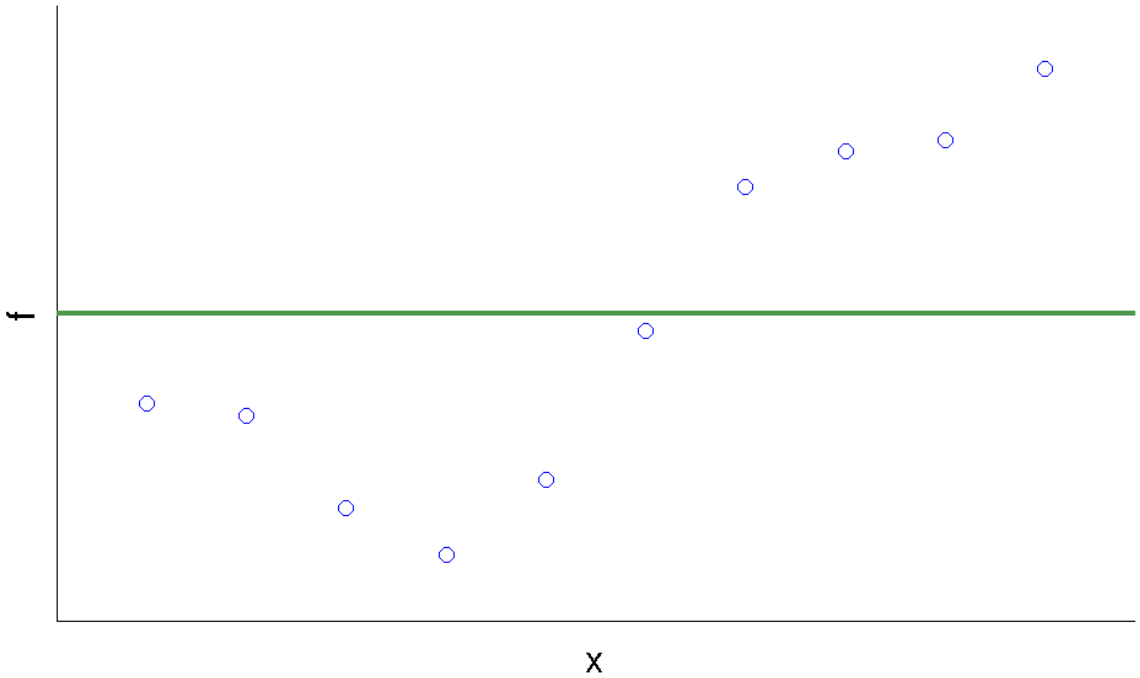
Parameterised directly by the training data

- Very flexible class of models with limited prior assumptions
- Infinite number of features/basis functions
- Finite, often very small, number of parameters left to deal with
- Intuitive, high level priors: smoothness, periodicity

Gaussian Processes

Prior distribution over $f(x)$ is Gaussian

$$\begin{aligned} f &\sim \mathcal{GP}(m,\ k) \\ \Rightarrow\ P(f(x)) &= \mathcal{N}(m(x),\ k(x)) \end{aligned} \tag{2}$$



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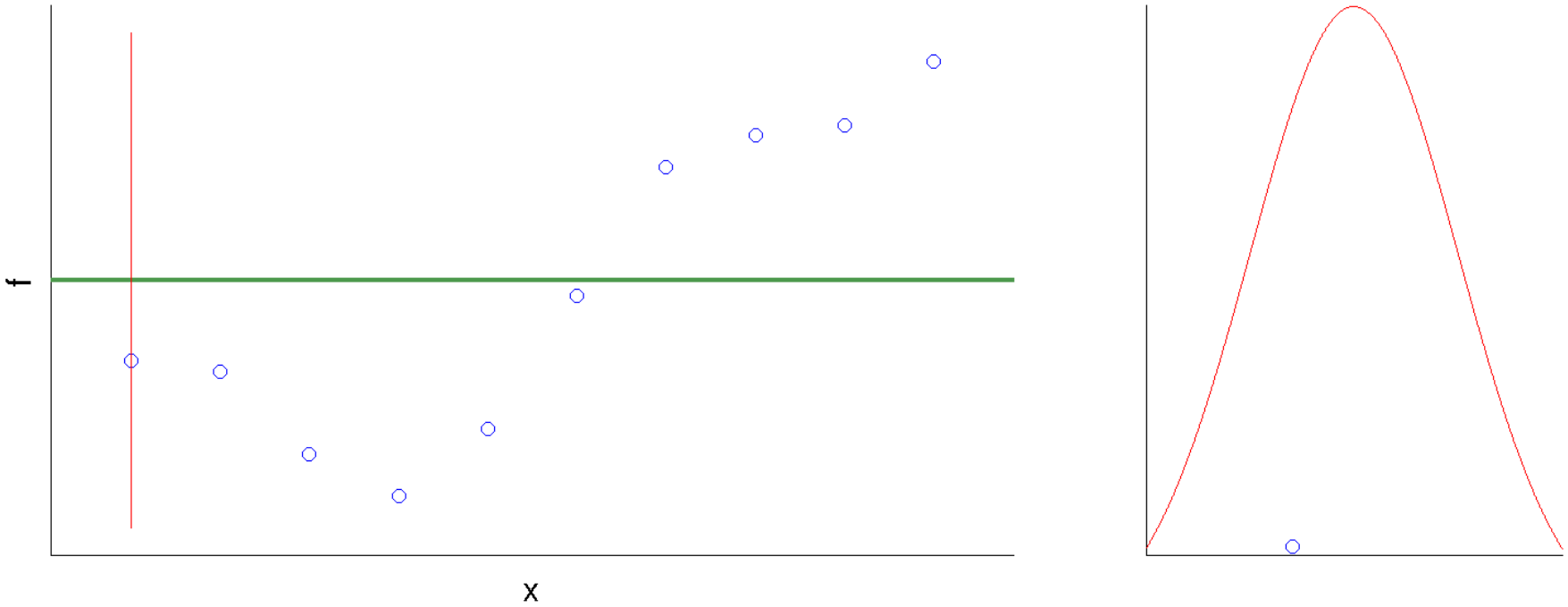

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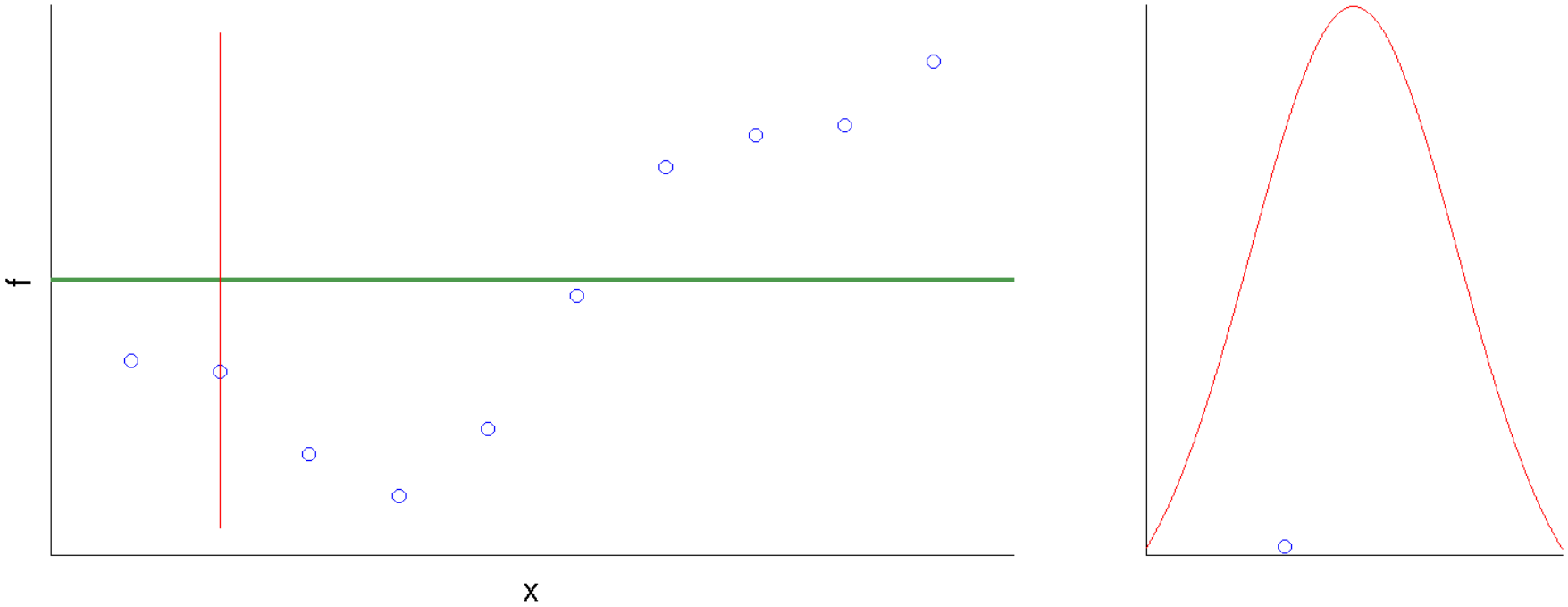

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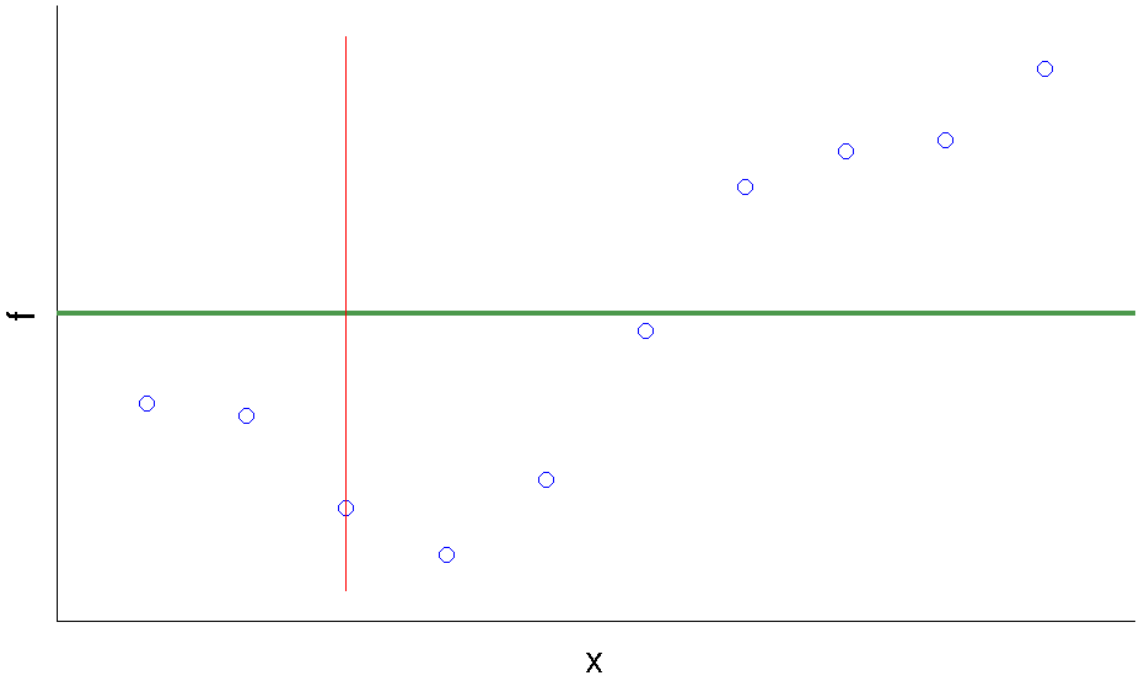

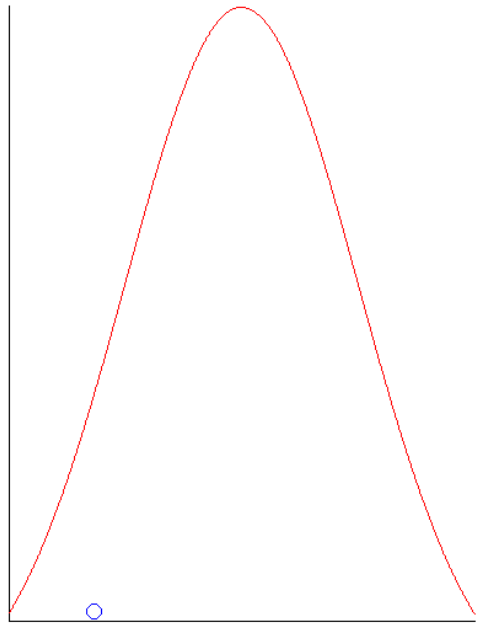

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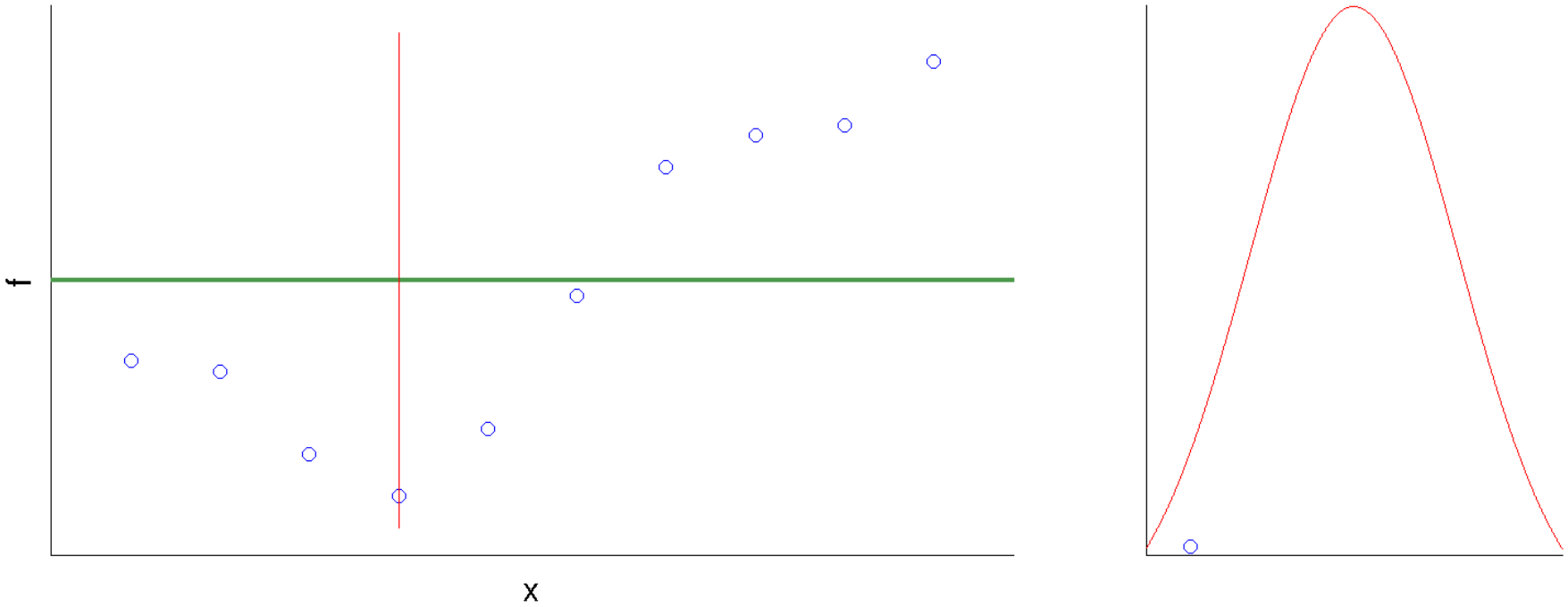

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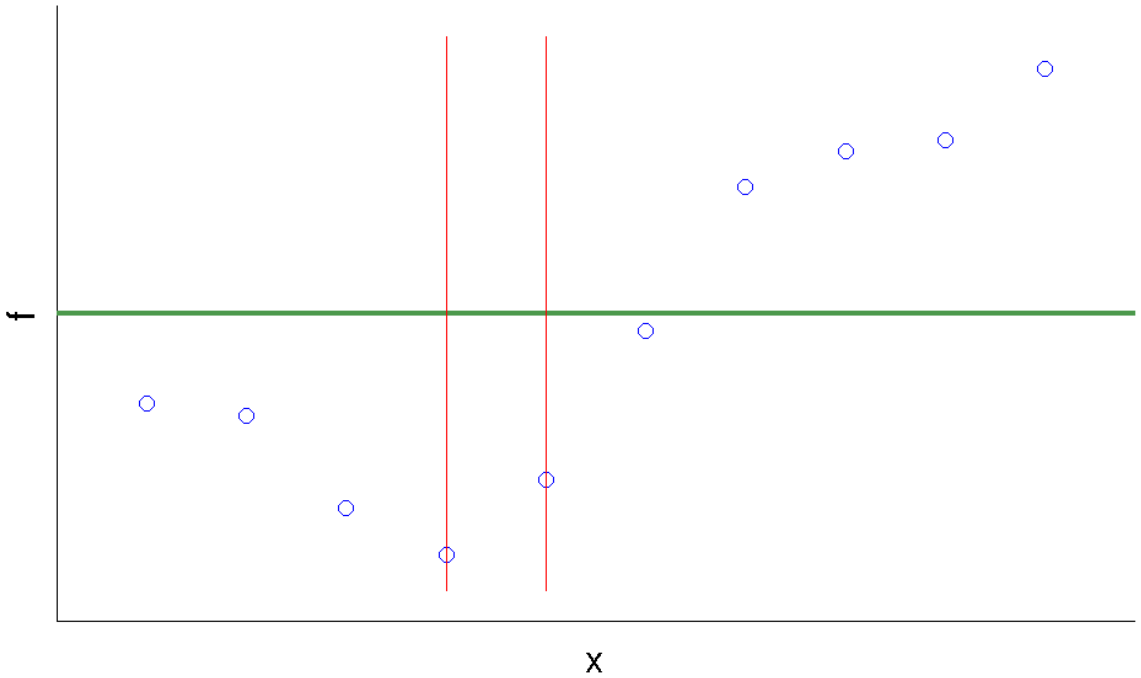

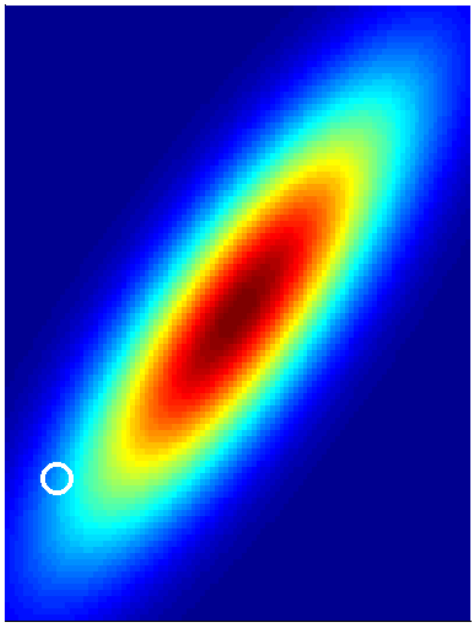

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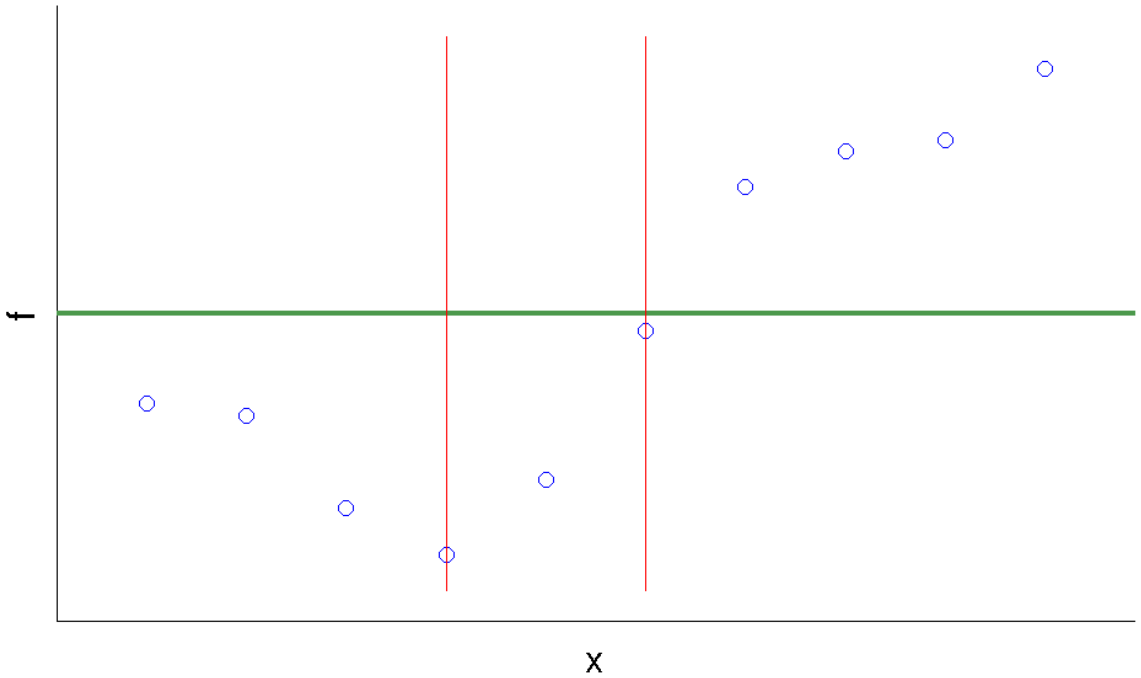

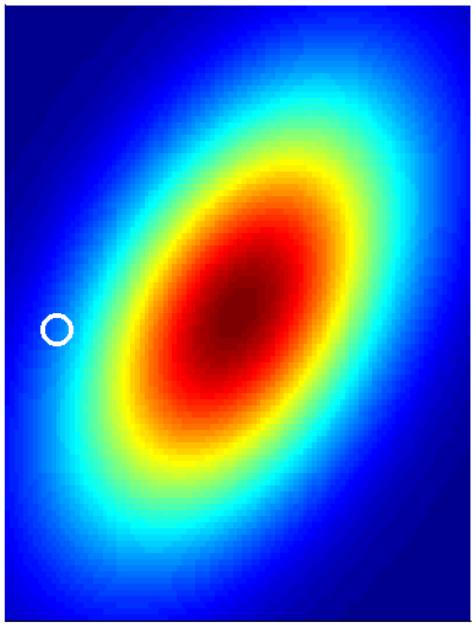

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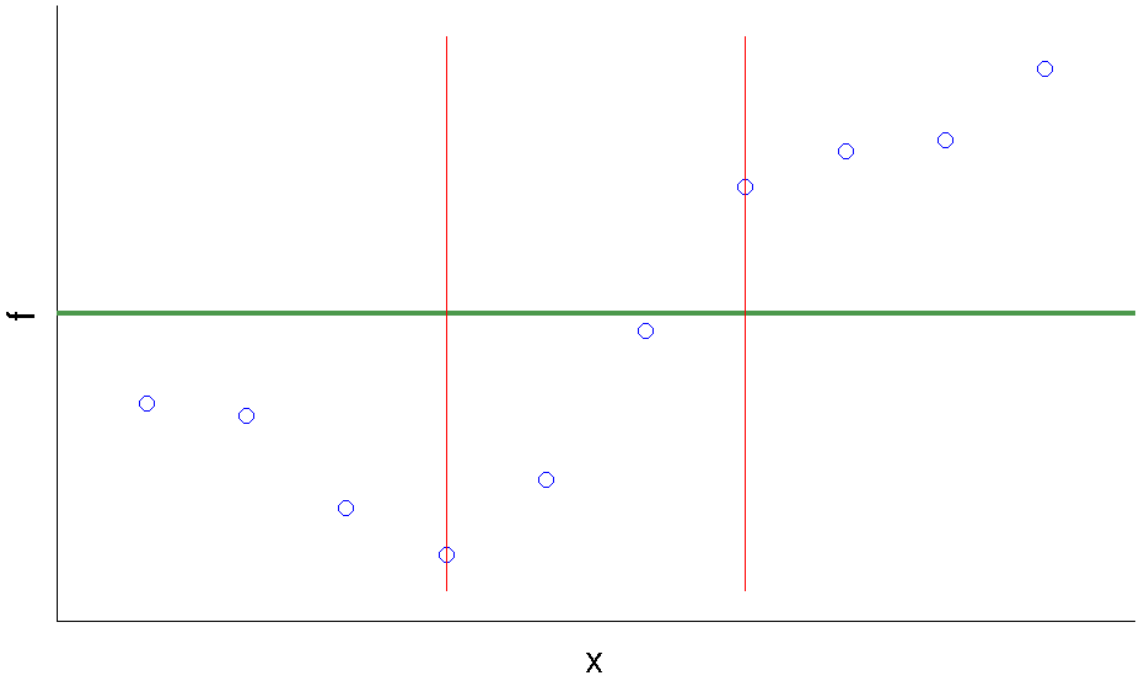

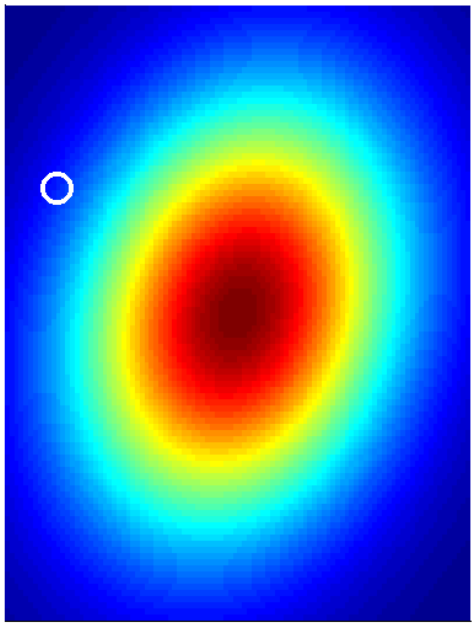

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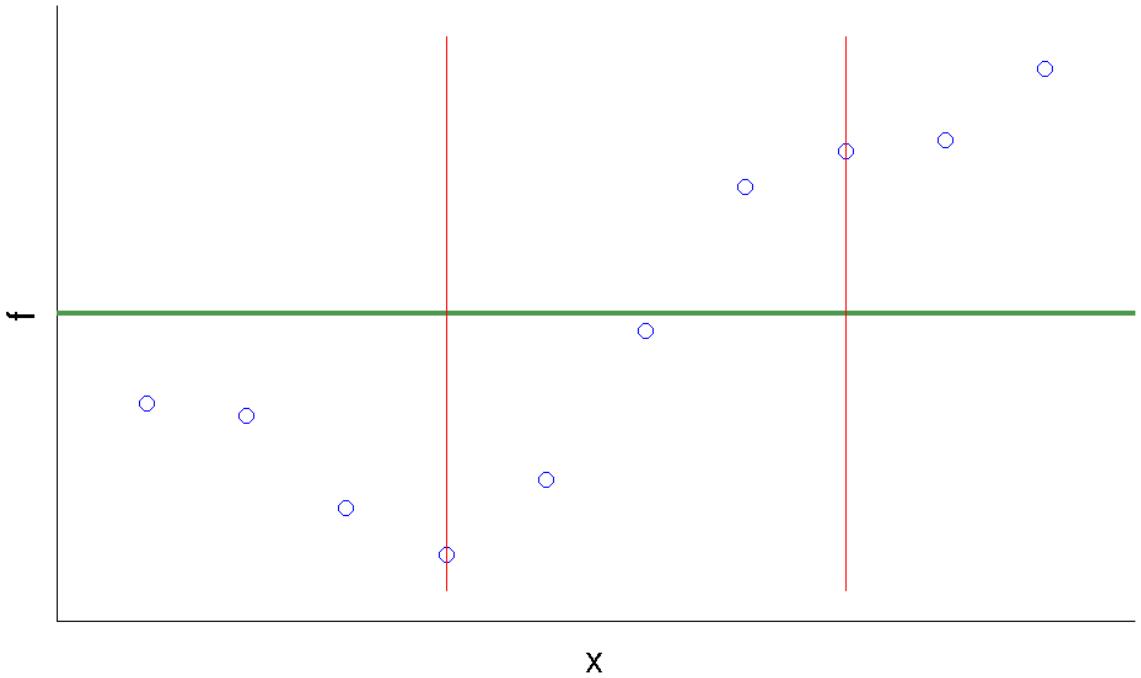

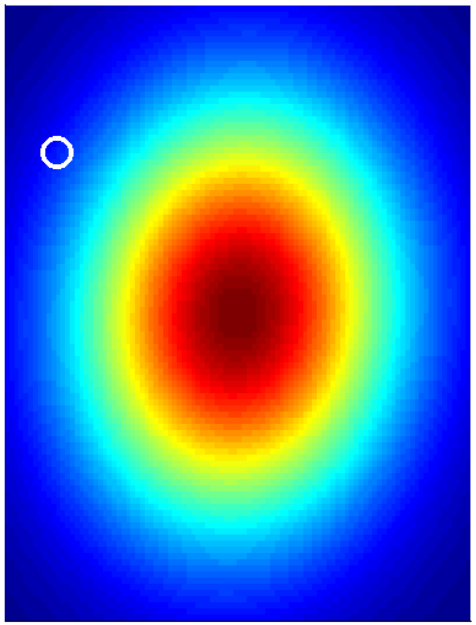

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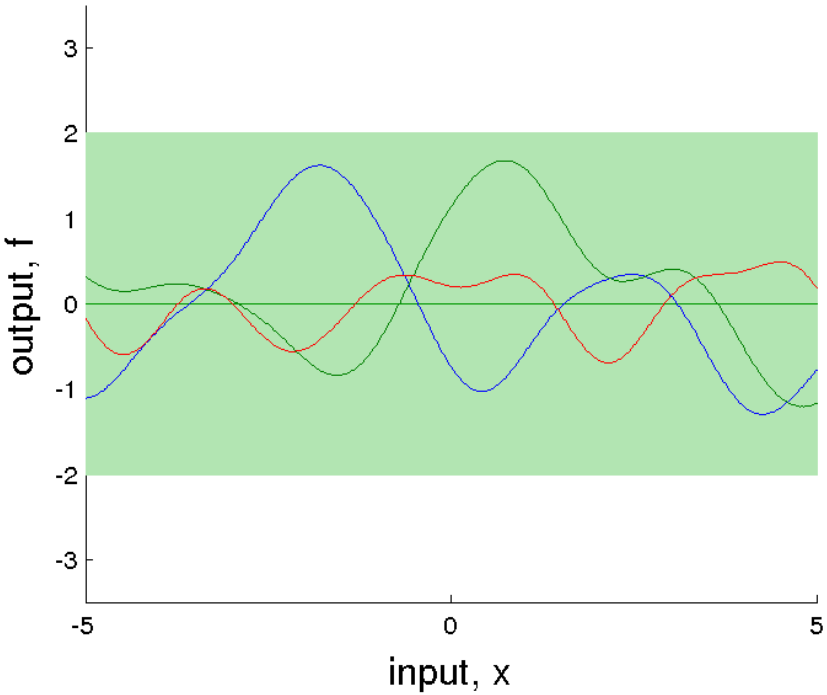
$$\begin{bmatrix} f \\ f_* \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m(x) \\ m(x_*) \end{bmatrix}, \begin{bmatrix} k(x,x) & k(x,x_*) \\ k(x_*,x) & k(x_*,x_*) \end{bmatrix}\right) \tag{3}$$



Prior, $P(f, f^*)$

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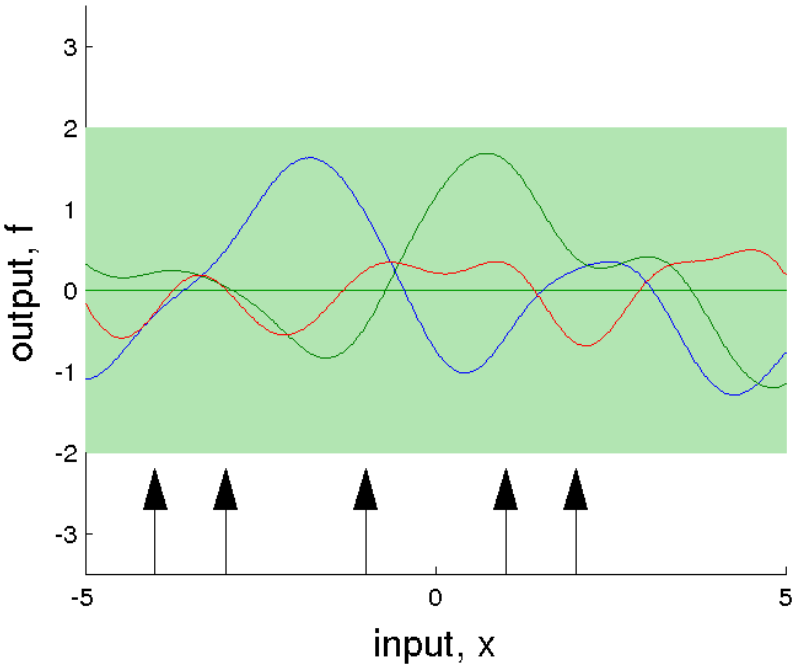

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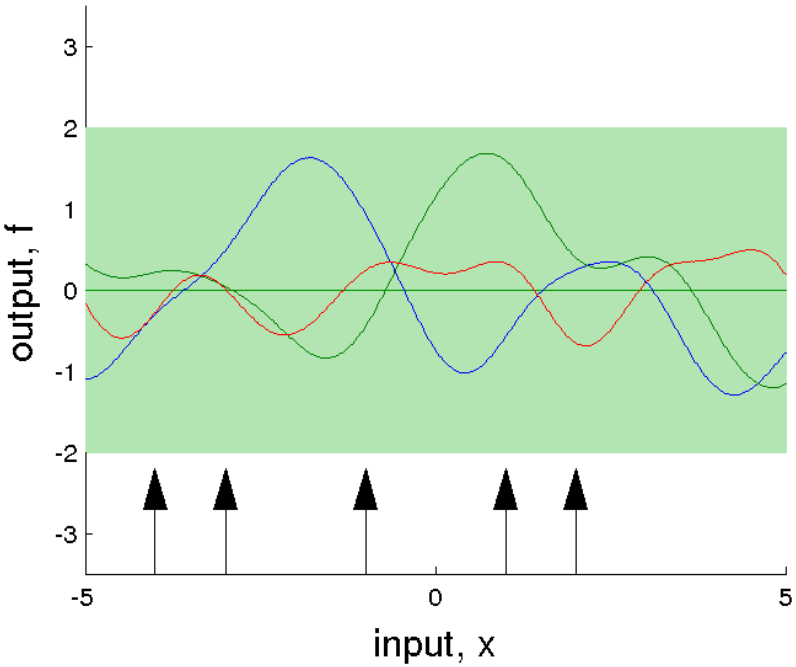

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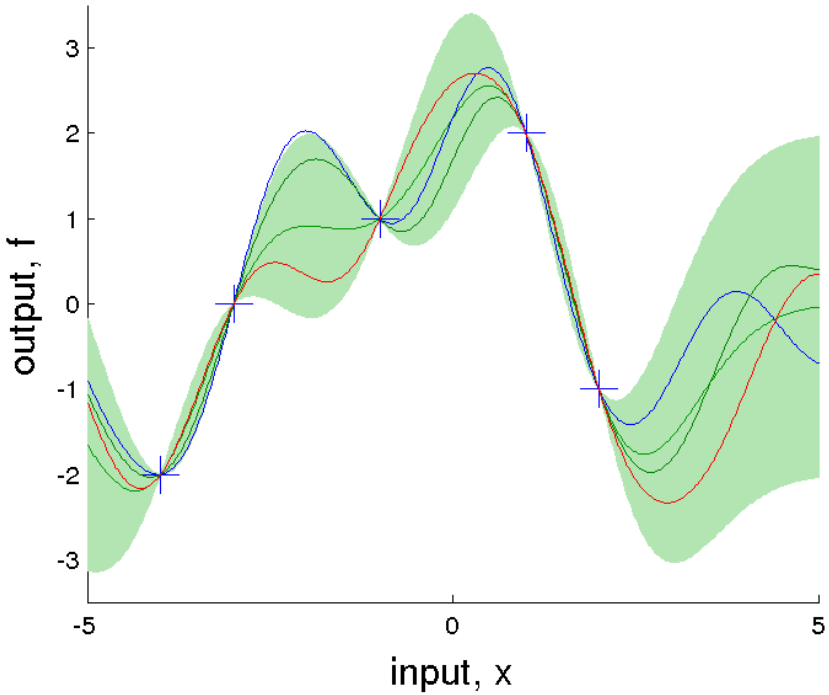

Posterior, $P(f^* \mid f)$

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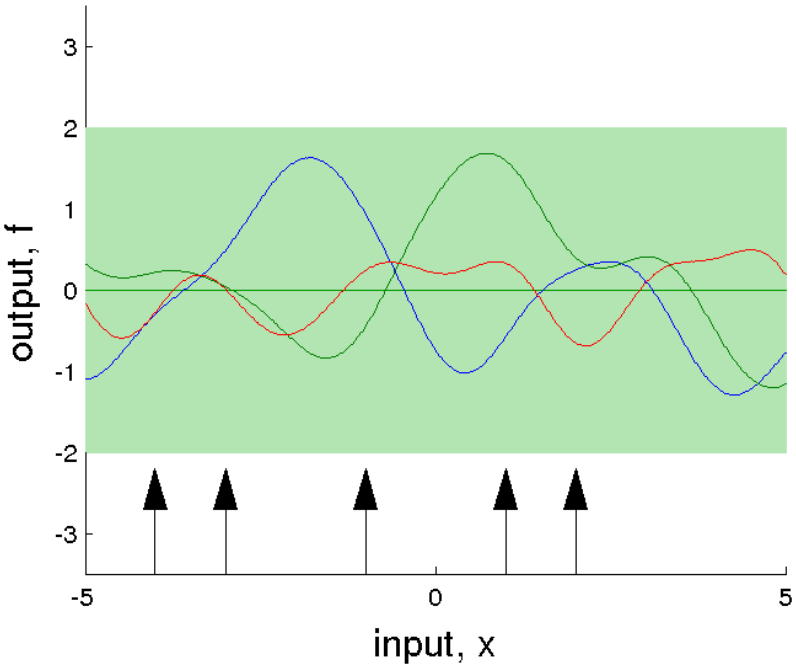

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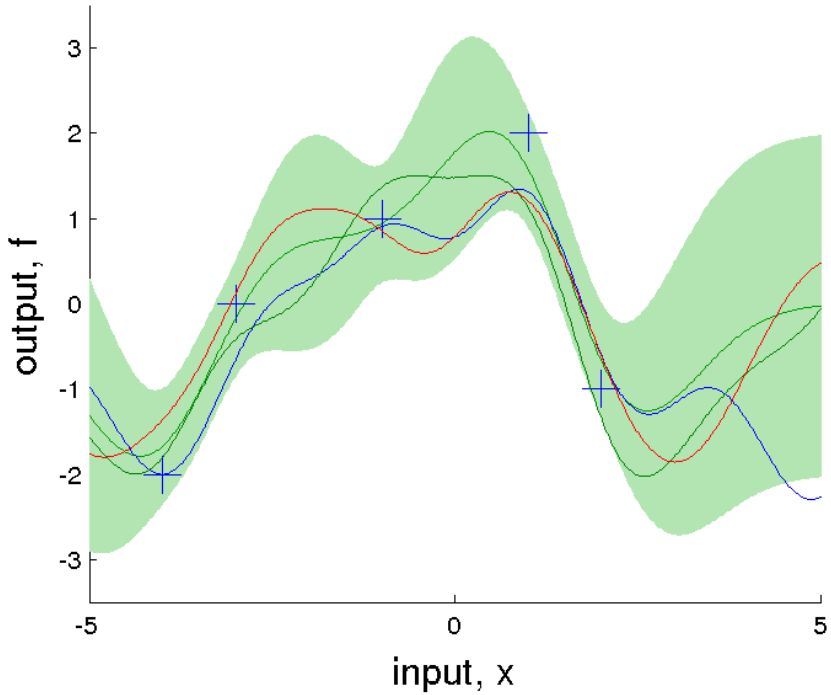
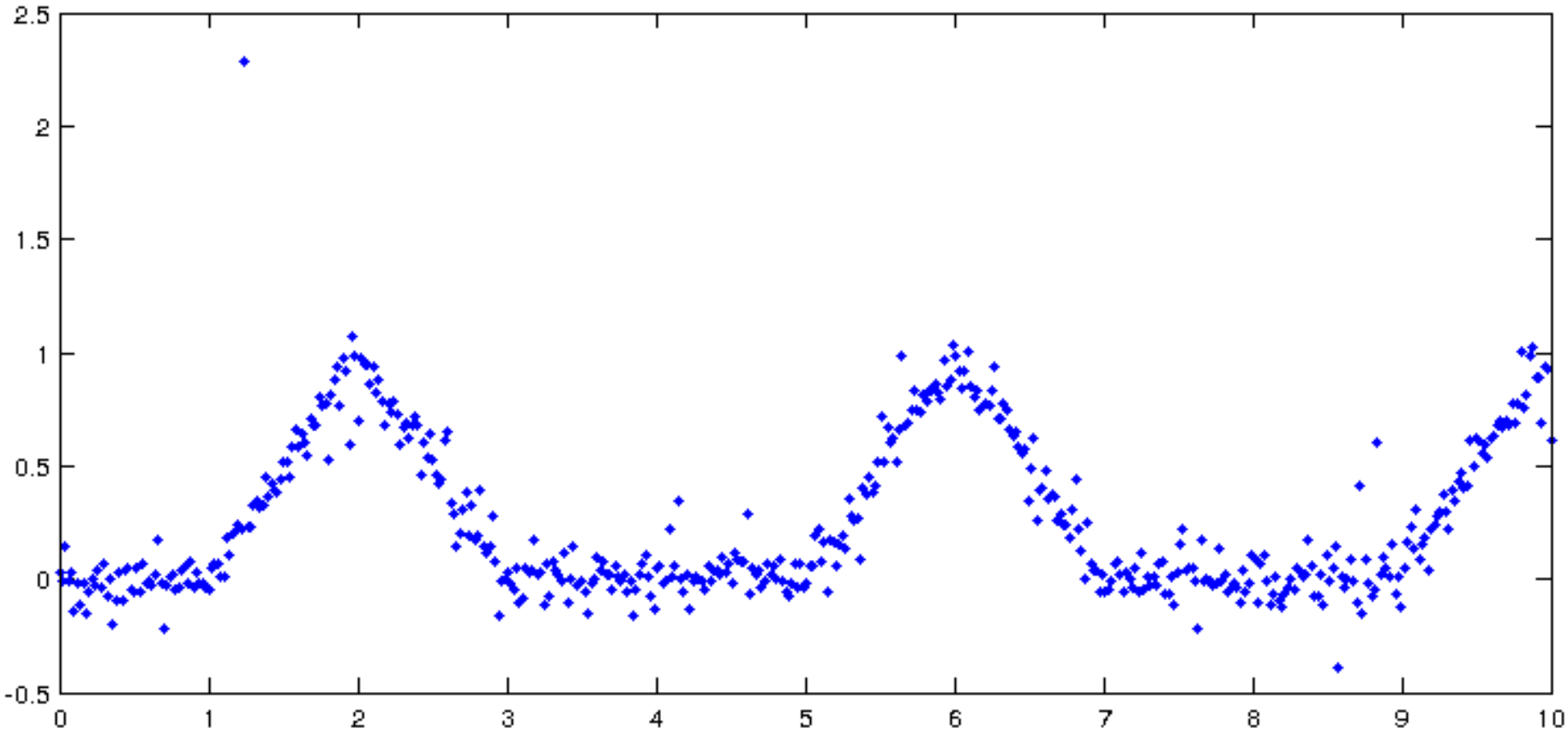


Posterior, $P(f^* \mid y)$

Gaussian Processes

- But my system doesn't have Gaussian noise...
- ...and my system isn't smooth



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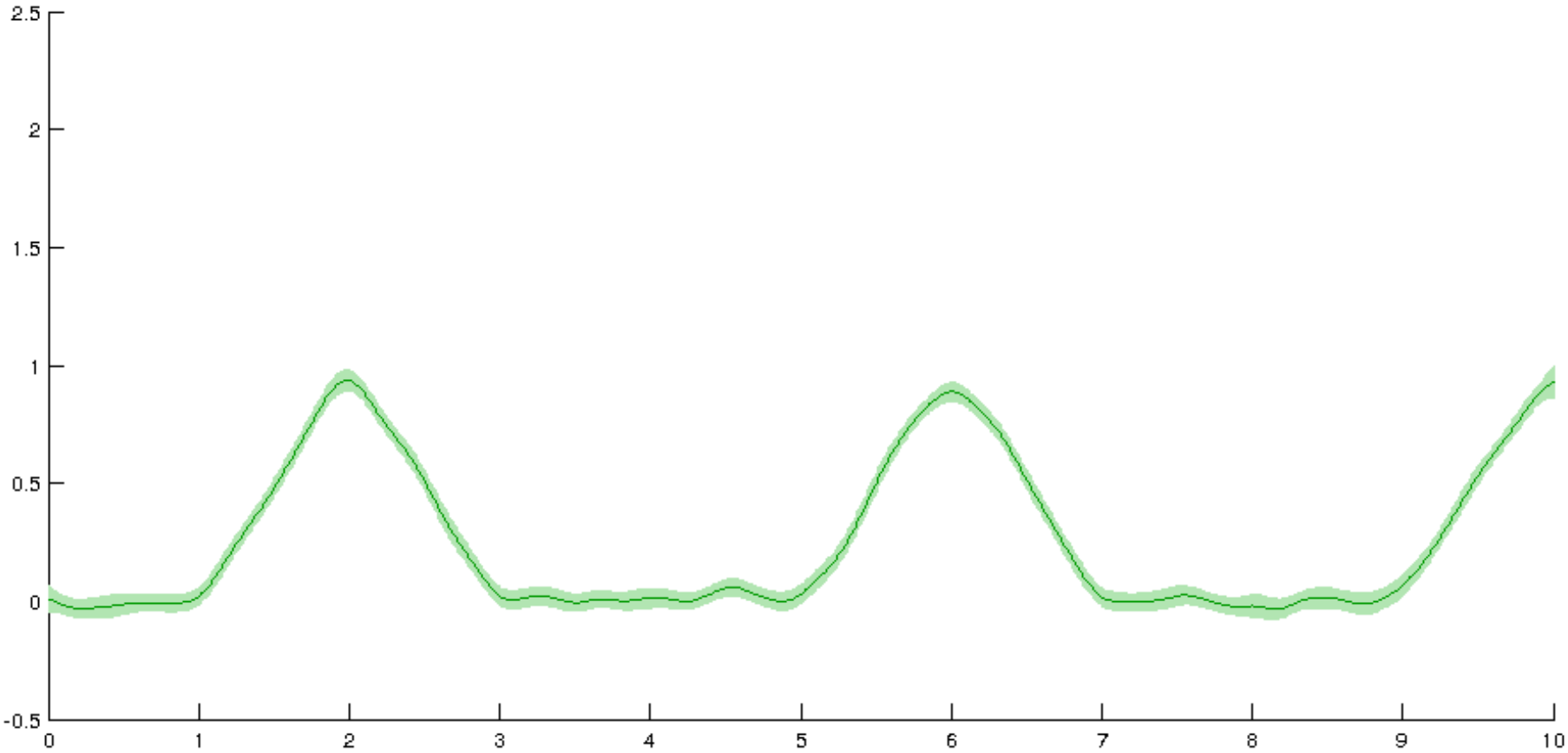

GP-NARX Models

Use a GP to model the mapping,

$$y_t = f(y_{t-1}, \ldots, y_{t-k}) + \epsilon \tag{5}$$

$$f \sim \mathcal{GP}(m,\ k) \tag{6}$$

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If ϵ is Gaussian,

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$$\mu = k(x^*, X) \left[K(X, X) + \sigma_\epsilon^2 I\right]^{-1} y \tag{8}$$
$$\Sigma = k(x^*, x^*) - k(x^*, X) \left[K(X, X) + \sigma_\epsilon^2 I\right]^{-1} k(X, x^*) \tag{9}$$

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Problem: noise in the inputs, "errors in the variables"

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- Noisy Input GP, [McHutchon & Rasmussen, NIPS 2011]

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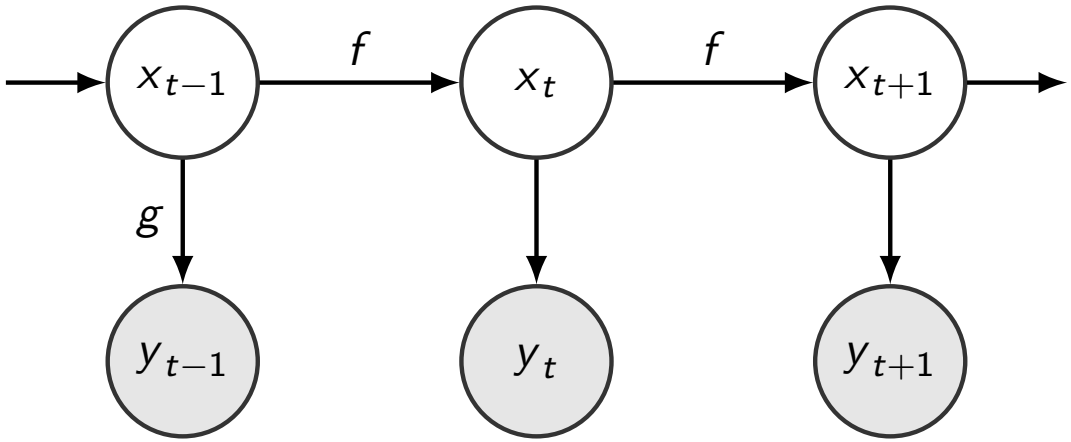
- GP-FNARX, [Frigola & Rasmussen, CDC 2013]

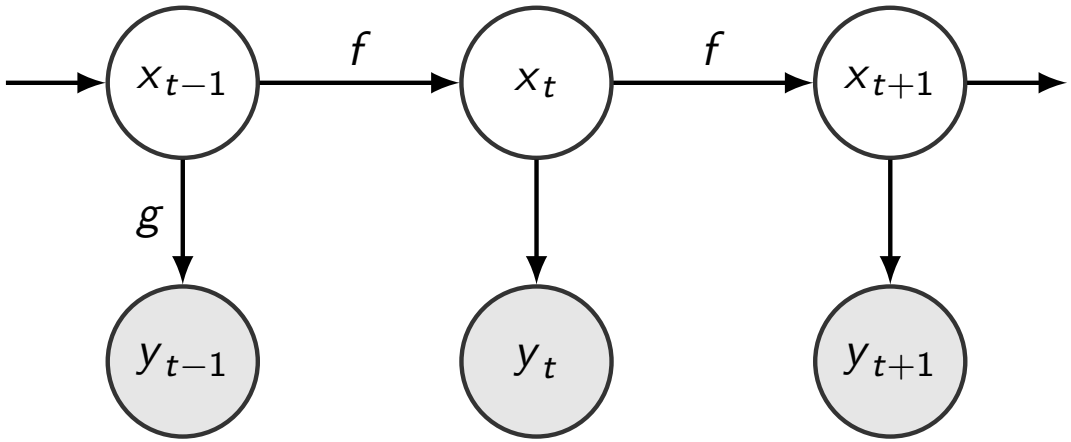
GP Latent Variable Models



GP-LVM [Lawrence, NIPS 2004], no latent state dynamics

- GP Dynamical Model, [Wang *et al.*, NIPS 2005 & PAMI 2008]
- Variational GP Dynamical System, [Damianou *et al.*, NIPS 2011]
- BayesFilter [Ko & Fox, 2009 & 2011]

GP Latent Variable Models



$$x_t = f(x_{t-1}) + \epsilon \tag{12}$$

$$y_t = g(x_t) + \nu \tag{13}$$

- $f \sim \mathcal{GP}(m, k)$

Analytic Approach

$$x_t = f(x_{t-1}) + \epsilon \tag{12}$$
$$y_t = g(x_t) + \nu \tag{13}$$

- $f \sim \mathcal{GP}(m,\ k)$
- $g = C$ or $g \sim \mathcal{GP}(m_g,\ k_g)$
- $\epsilon \sim \mathcal{N}(0, \Sigma_\epsilon)$ and $\nu \sim \mathcal{N}(0, \Sigma_\nu)$

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- Introduce a pseudo-training set $\{\tilde{x},\ \tilde{f}\}$

$$\theta = \{\tilde{x},\ \tilde{f},\ \theta_m,\ \theta_k,\ C,\ \Sigma_\epsilon,\ \Sigma_\nu\} \tag{14}$$

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- Treat f in a Bayesian way, and θ in a maximum likelihood manner

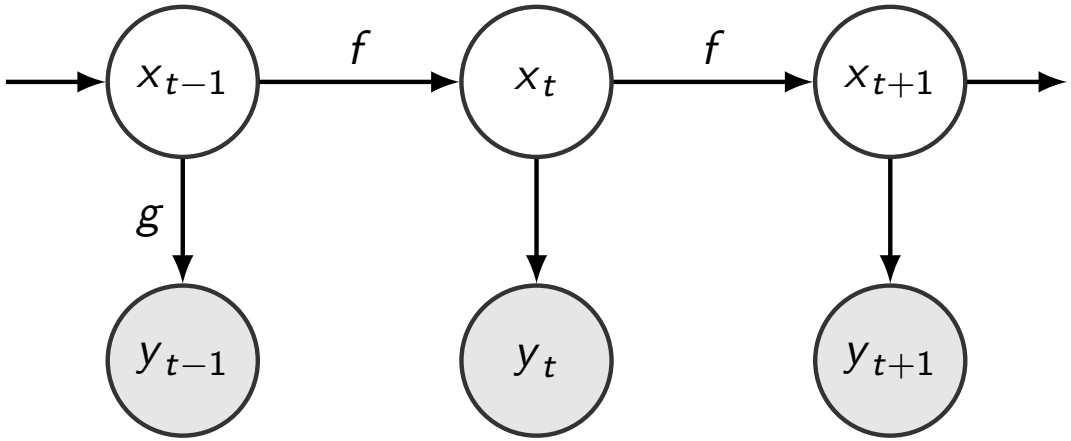
Marginal Likelihood

Probability of the data under the model averaging over our belief about the latent variables and the function f

$$P(y_1, \ldots, y_T \mid \theta) = P(y_1 \mid \theta)P(y_2 \mid y_1, \theta)P(y_3 \mid y_2, y_1, \theta)\ldots \quad (15)$$

$$= P(y_1 \mid \theta)\prod_{t=2}^{T} P(y_t \mid y_{t-1}, \ldots, y_1, \theta) \quad (16)$$

Marginal Likelihood



$$P(y_t \mid y_{t-1}, \ldots, y_1, \theta) = \int \int \underbrace{P(x_{t-1} \mid y_{t-1}, \ldots, y_1)}_{\text{filtered state at t-1}} \underbrace{P(x_t \mid x_{t-1})}_{\text{transition}} \underbrace{P(y_t \mid x_t)}_{\text{observation}} dx_t dx_{t-1} \quad (17)$$

Direct Optimisation

- Project filtered state distribution to a Gaussian
- Method 1:

$$\int P(y_t \mid x_t) \int P(x_{t-1} \mid y_{t-1}, \ldots, y_1)\, P(x_t \mid x_{t-1})\, dx_{t-1}\, dx_t$$

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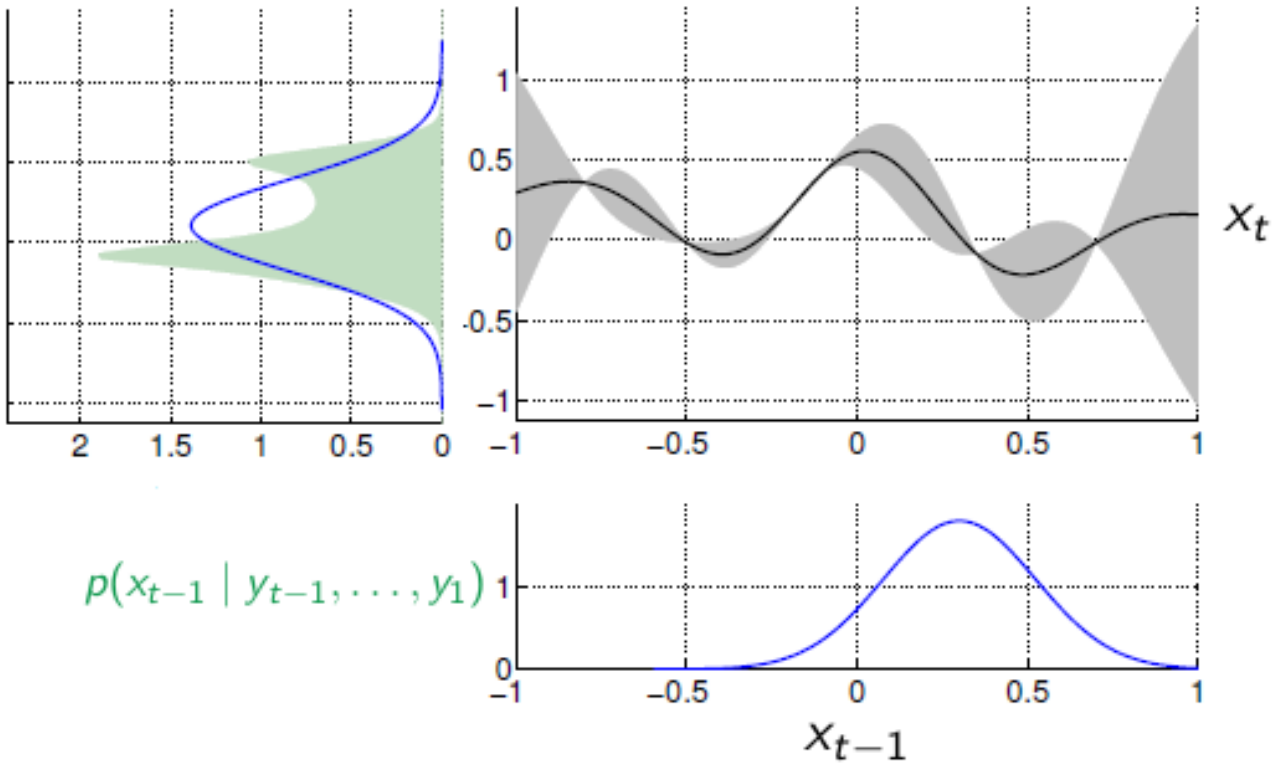


Direct Optimisation

- Project filtered state distribution to a Gaussian
- Method 1: Project $P(x_t \mid y_{t-1}, \ldots, y_1)$ to a Gaussian

$$\int P(y_t \mid x_t) \int P(x_{t-1} \mid y_{t-1}, \ldots, y_1)\, P(x_t \mid x_{t-1})\, dx_{t-1}\, dx_t$$

- Method 2:

$$\int P(x_{t-1} \mid y_{t-1}, \ldots, y_1) \int P(x_t \mid x_{t-1})\, P(y_t \mid x_t)\, dx_t\, dx_{t-1}$$

Direct Optimisation

- Project filtered state distribution to a Gaussian
- Method 1: Project $P(x_t \mid y_{t-1}, \ldots, y_1)$ to a Gaussian

$$\int P(y_t \mid x_t) \int P(x_{t-1} \mid y_{t-1}, \ldots, y_1)\, P(x_t \mid x_{t-1})\, dx_{t-1}\, dx_t$$

- Method 2: Simple sampling

$$\int P(x_{t-1} \mid y_{t-1}, \ldots, y_1) \int P(x_t \mid x_{t-1})\, P(y_t \mid x_t)\, dx_t\, dx_{t-1}$$

- Similar steps to compute filter distribution at time t, $P(x_t \mid y_t, \ldots, y_1)$

Direct Optimisation

for t = 1:T

compute

$P(x_t \mid y_t, \ldots, y_1, \theta)$

$\log P(y_t \mid y_{t-1}, \ldots, y_1, \theta)$

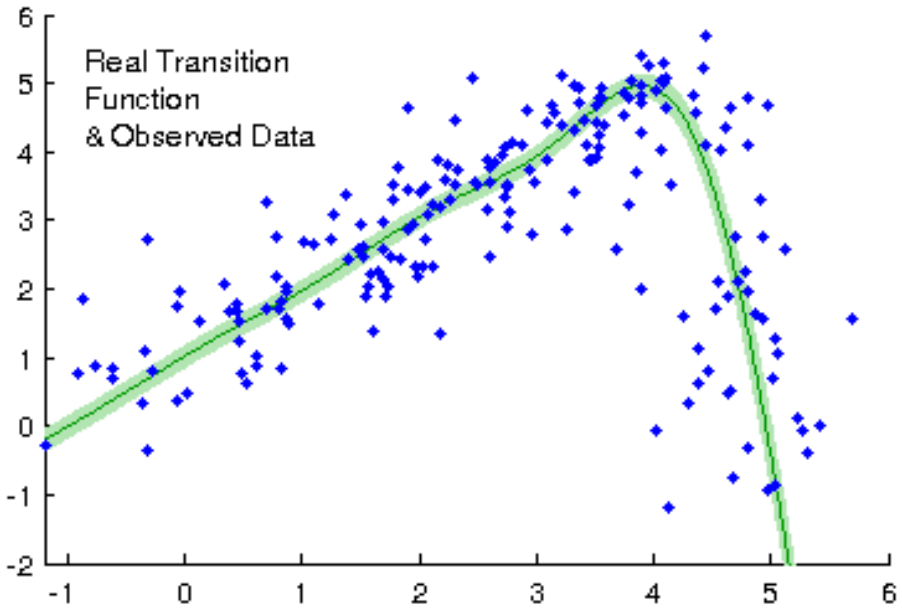
derivatives w.r.t. θ

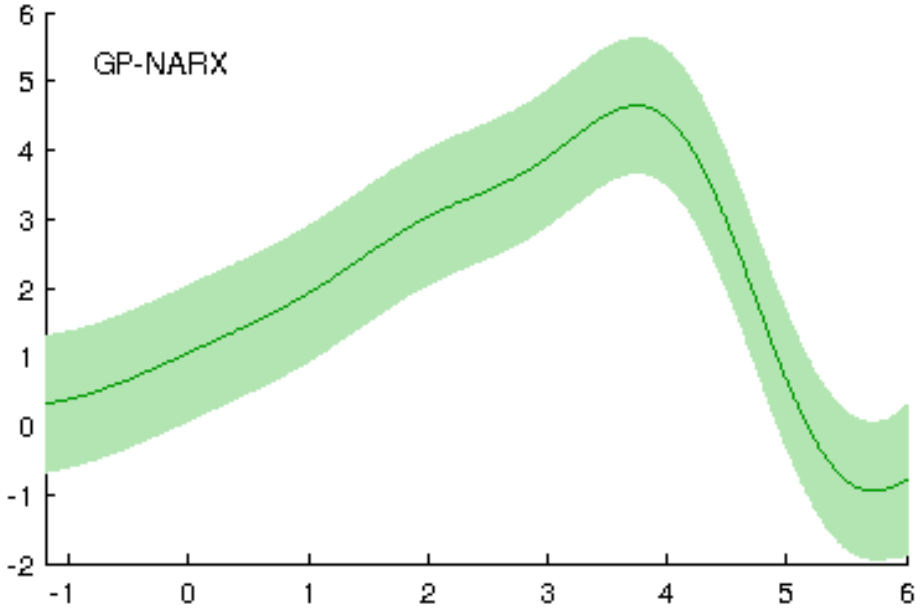
end

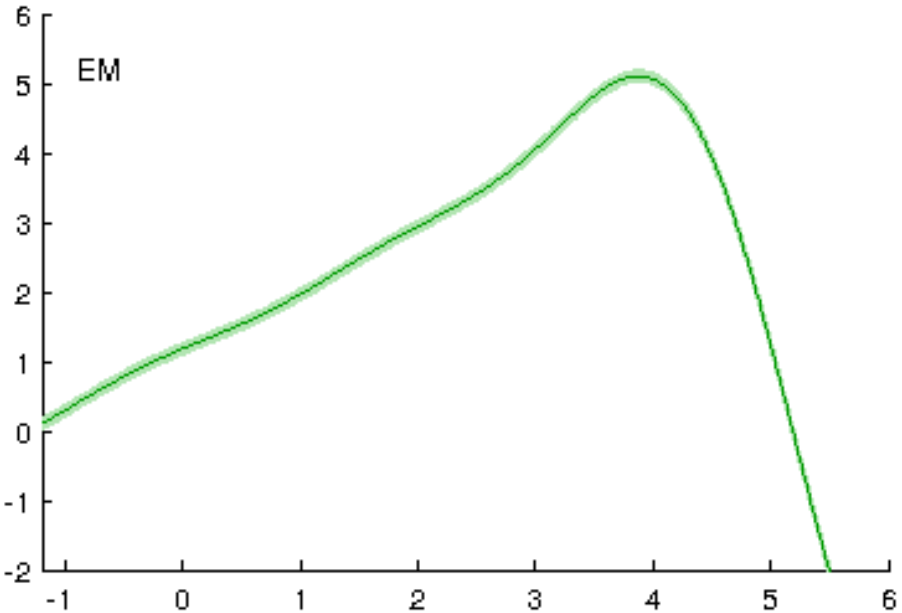
$$\log P(y_1, \ldots, y_T \mid \theta) = \sum \log P(y_t \mid y_{t-1}, \ldots, y_1, \theta)$$

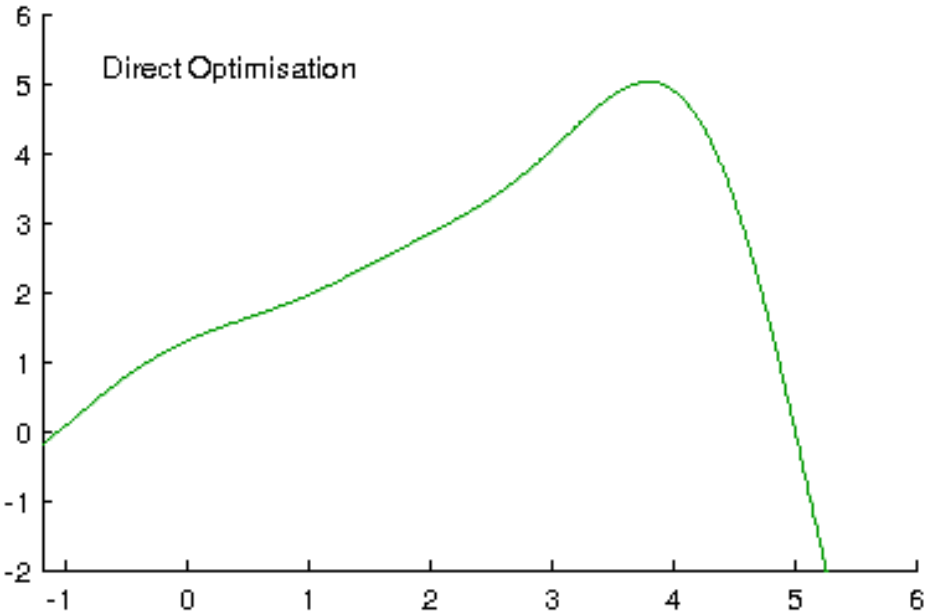
optimise w.r.t. θ

Example: 1D System

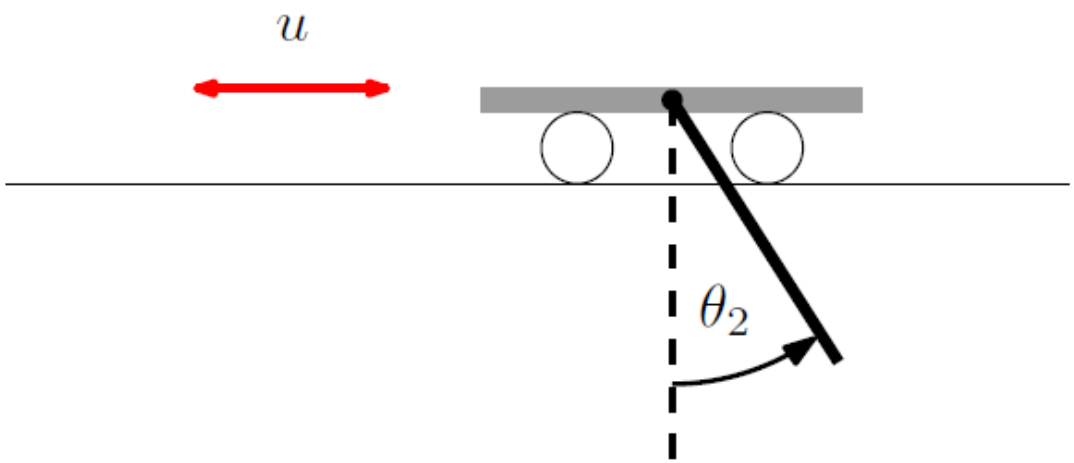






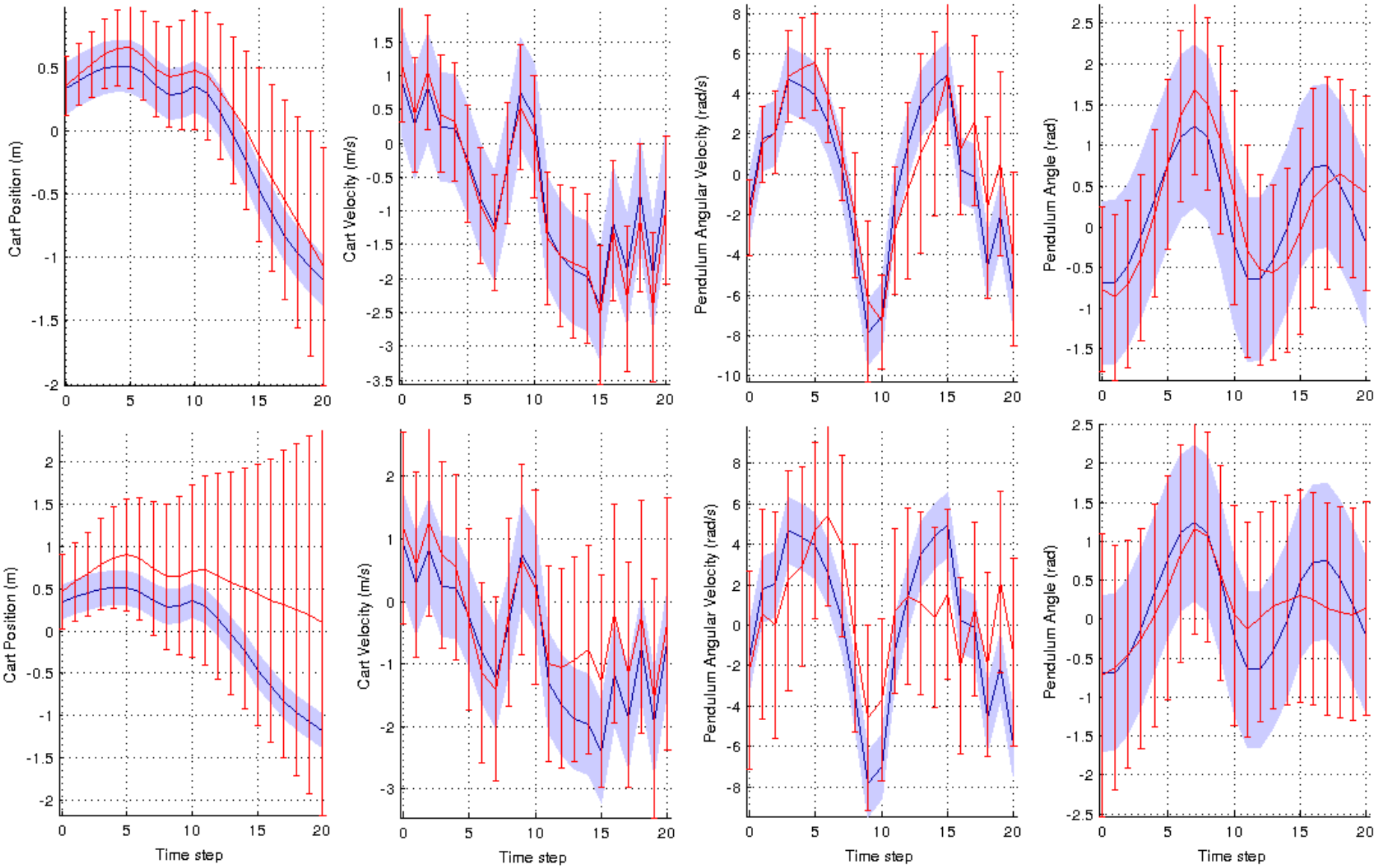


Example: Cart and Pole System



- 4 state variables
- 1 control - driven by white noise
- find distribution over future trajectory given feed-forward controls

Example: Cart and Pole System



System Identification with GP-SSMs using particle Markov Chain Monte Carlo (PMCMC) [1]

Nonlinear state-space model:

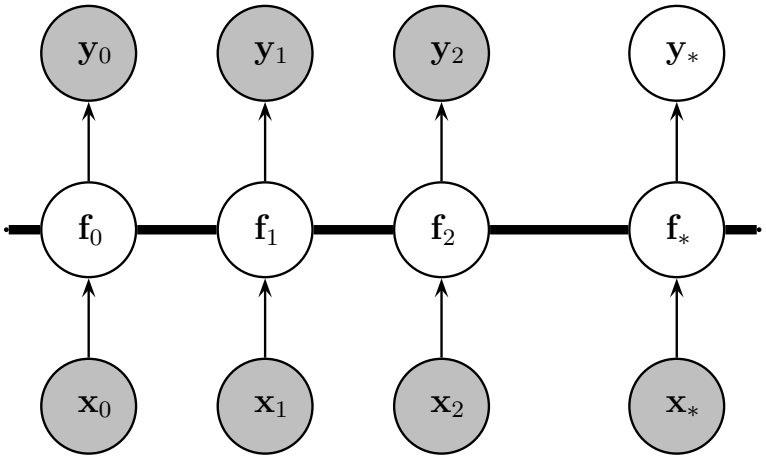
$$x_{t+1} = f(x_t) + \mathbf{v}_t,$$
$$\mathbf{y}_t = g(x_t) + \mathbf{e}_t.$$

Generative model for Gaussian process state-space models (GP-SSMs):

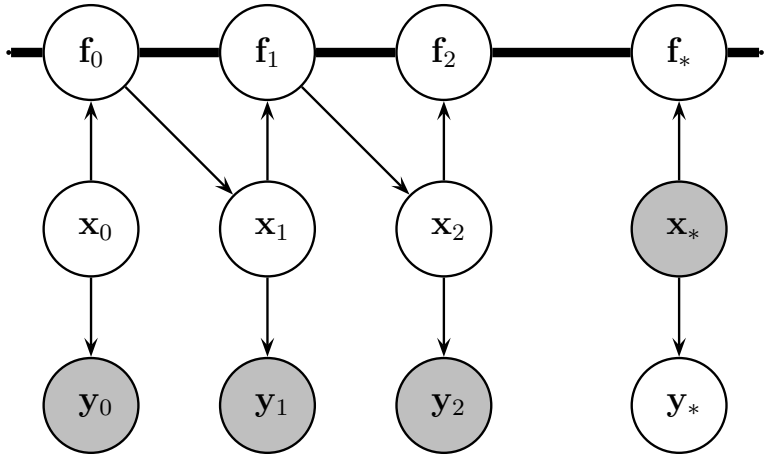
$$f(x_t) \sim \mathcal{GP}\big(m_{\theta_x}(x_t), k_{\theta_x}(x_t, x_t')\big),$$
$$x_{t+1} \mid \mathbf{f}_t \sim \mathcal{N}(x_{t+1} \mid \mathbf{f}_t, \mathbf{Q}),$$
$$\mathbf{y}_t \mid x_t \sim p(\mathbf{y}_t \mid x_t, \theta_{\mathbf{y}}).$$

[1] joint work with Fredrik Lindsten, Thomas B. Schön and Carl Rasmussen, NIPS 2013

Graphical models



Gaussian process regression, $\mathbf{y} = f(x) + \epsilon$



GP-SSM

Computing the smoothing distribution is system identification!

This is a very particular property of this model!!!

Challenge: find a good representation of

$$p(x_{0:T} \mid \theta, \mathbf{y}_{0:T}).$$

Sampling the states given the data (smoothing)

Posterior is proportional to likelihood times prior

$$p(\mathbf{x}_{0:T} \mid \mathbf{y}_{0:T}, \theta) \propto p(\mathbf{y}_{0:T} \mid \mathbf{x}_{0:T}, \theta)\ p(\mathbf{x}_{0:T} \mid \theta).$$

We use Particle Gibbs with Ancestor Sampling (PG-AS).

(cf. Thomas Schön's tutorial)

Marginalising the latent function

To do inference we need the joint distribution over latent variables

$$p(\mathbf{x}_{0:T}, \mathbf{f}_{0:T} \mid \theta)$$

But $\mathbf{x}_{0:T}$ and $\mathbf{f}_{0:T}$ are tightly coupled. We can marginalise the latent function analytically

$$p(x_{1:T} \mid \theta, x_0) = \prod_{t=1}^{T} p(x_t \mid \theta, x_{0:t-1}) = \prod_{t=1}^{T} \mathcal{N}\big(x_t \mid \mu_t(x_{0:t-1}), \Sigma_t(x_{0:t-1})\big).$$

where μ_t and Σ_t have analogous expressions to those in GP regression.

What about the hyper-parameters?

We have shown how PMCMC can give us samples from

$$p(x_{0:T} \mid \theta, \mathbf{y}_{0:T}).$$

We can alternate this with sampling from

$$p(\theta \mid x_{0:T}, \mathbf{y}_{0:T})$$

using slice sampling.

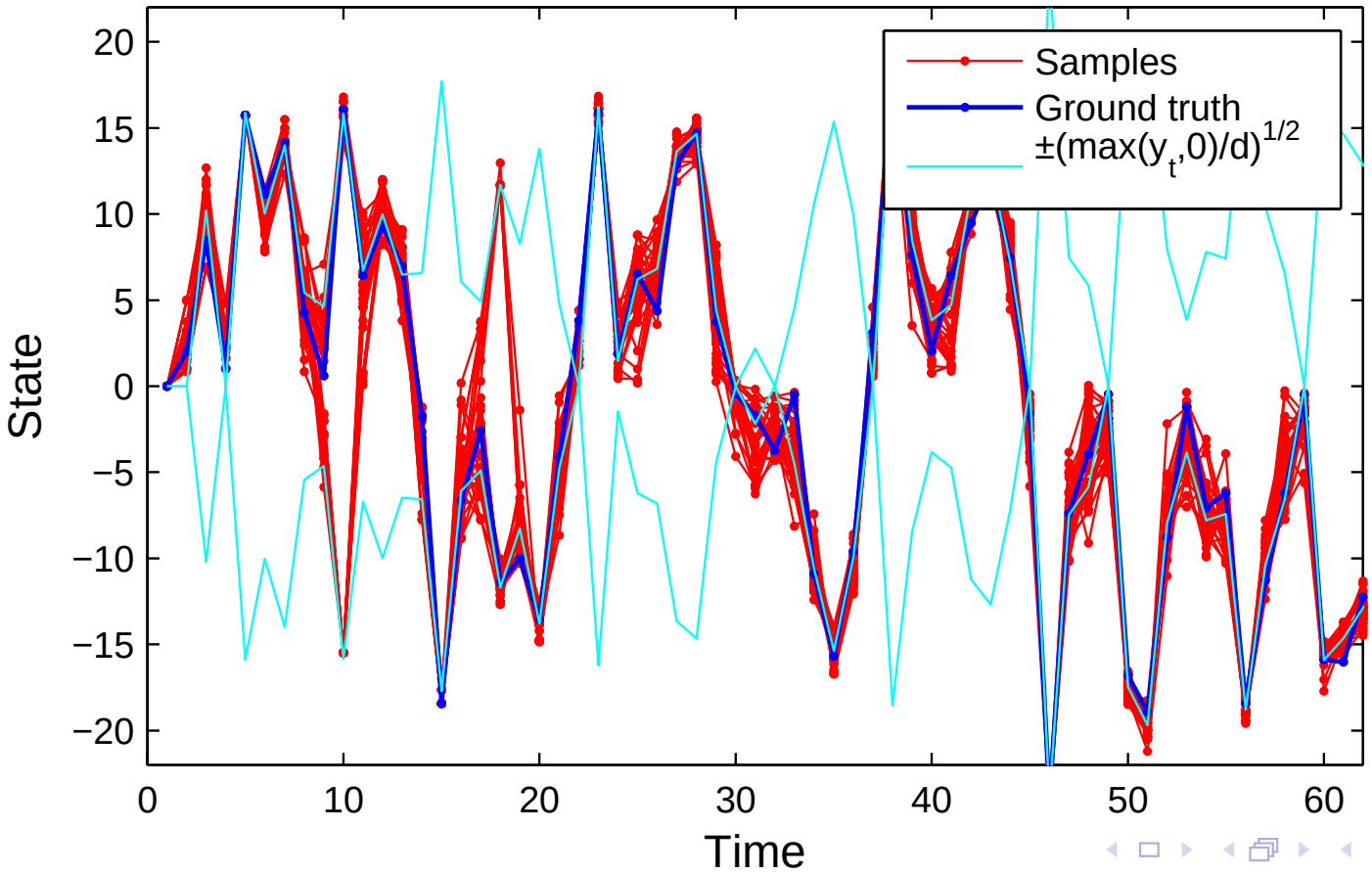
This is a (blocked) Gibbs sampler which ultimately targets

$$p(x_{0:T}, \theta \mid \mathbf{y}_{0:T}).$$

Experiments: 1-dim Benchmark System

$$x_{t+1} = ax_t + b\frac{x_t}{1+x_t^2} + cu_t + v_t, \quad v_t \sim \mathcal{N}(0, q),$$

$$y_t = dx_t^2 + e_t, \quad e_t \sim \mathcal{N}(0, r).$$



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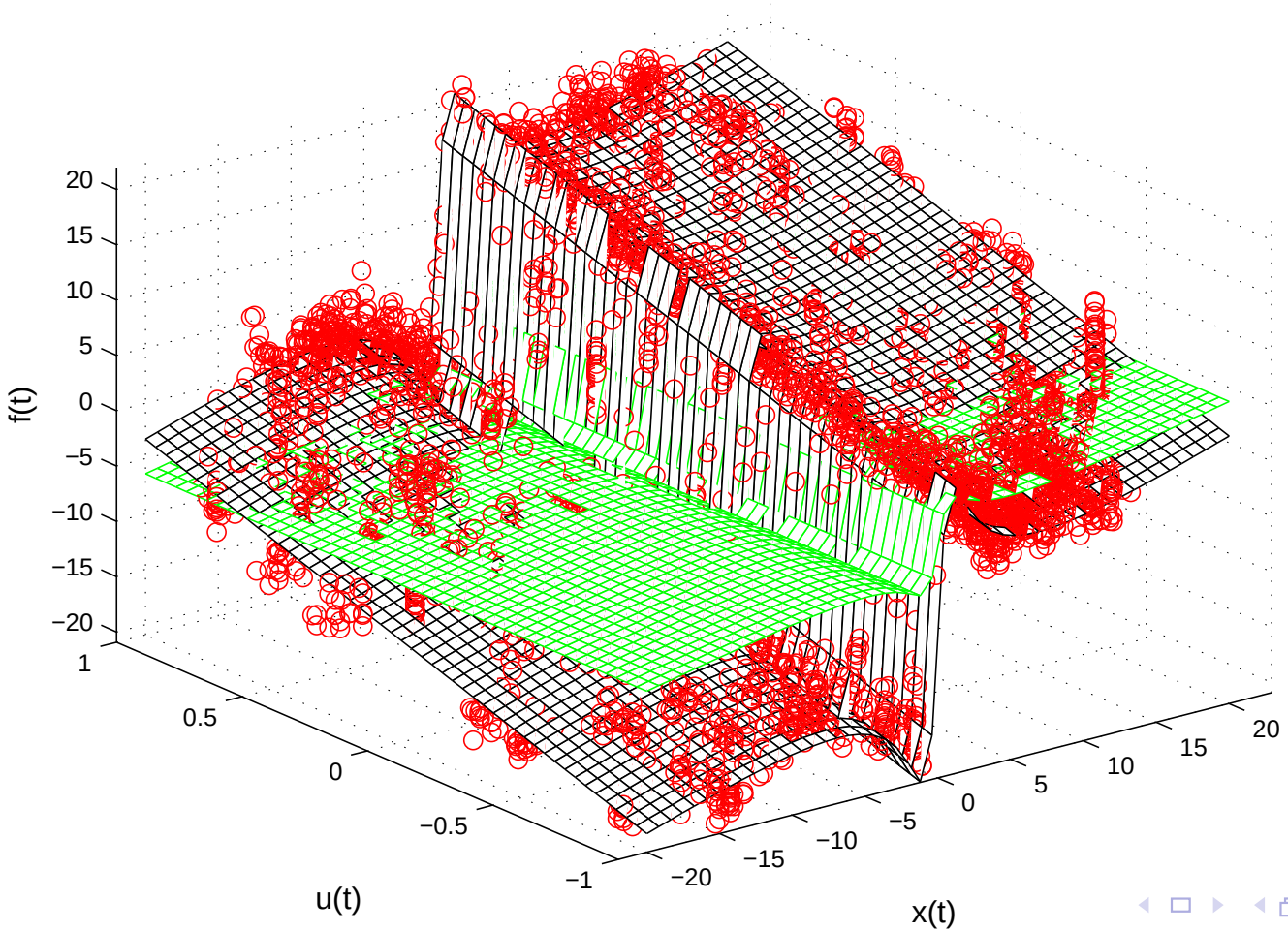
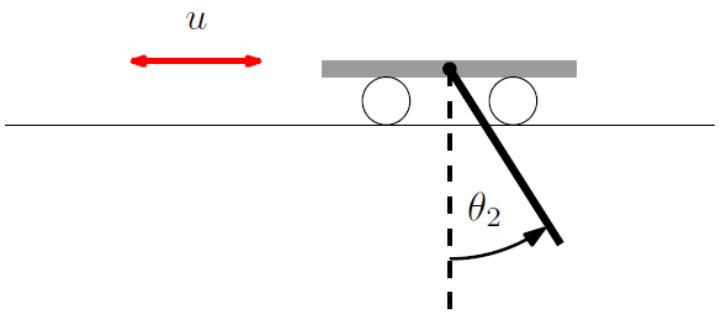
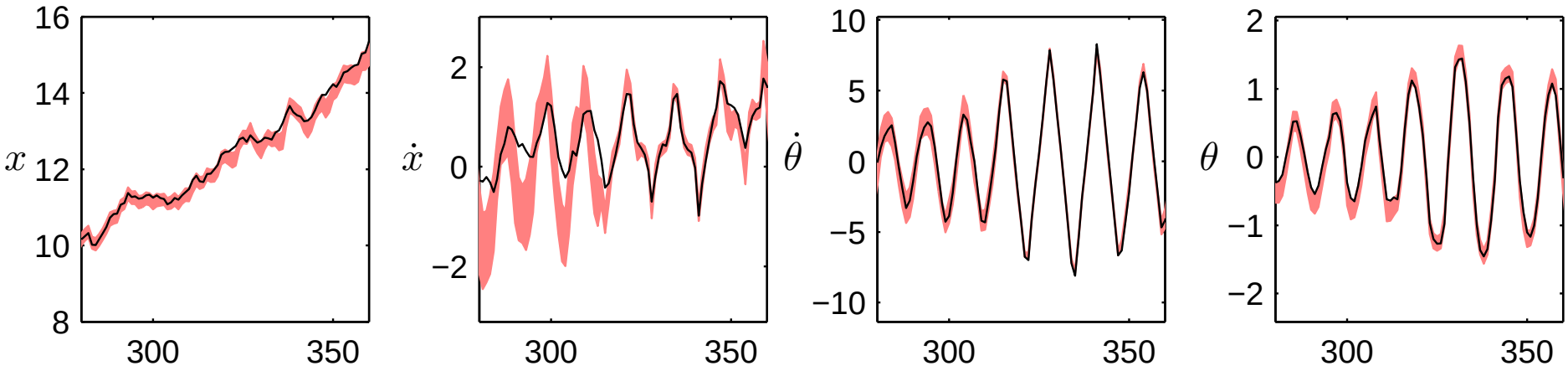


Experiments: 4-dim Cart and Pole System



One step ahead predictions with error bars



Take-home Messages

Bayesian inference:

- Quantifies uncertainty.
- Uses this uncertainty to make predictions/decisions.

Bayesian nonparametrics:

- Highly flexible - doesn't exclude complex explanations a priori.