Gaussian Process Models for Nonlinear Time Series

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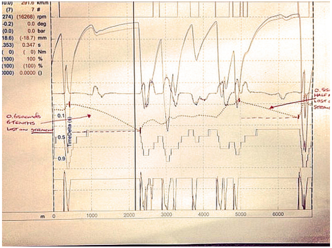
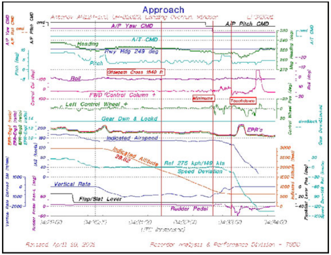
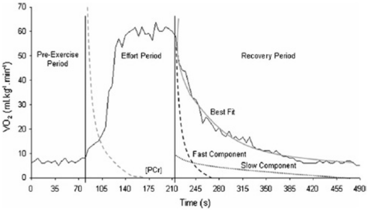
Outline

- Time series models.
- Bayesian & nonparametric & nonlinear.
- A zoo of GP-based models.
- Predictions in GP state-space models.
- ...

Time Series Models

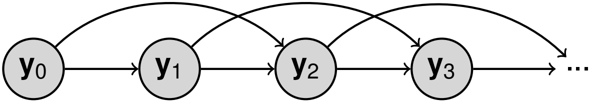






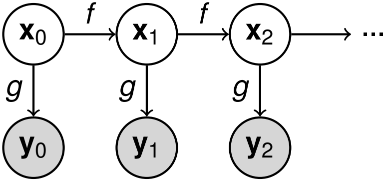
Classic Models of Time Series

Auto-regressive model (AR, ARX, NARX...)



$$\mathbf{y}_t = f(\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-n_y}) + \delta_t.$$

State-space models (SSM)



$$\mathbf{x}_{t+1} = f(\mathbf{x}_t) + \epsilon_t,$$
$$\mathbf{y}_t = g(\mathbf{x}_t) + \nu_t.$$

Bayesian

Model uncertainty.

Controlled overfitting.

No need to artificially limit the complexity of the models. There is no statistical price to pay for adding more parameters.

Nonparametric

Flexible.

Data is not condensed into a finite set of parameters.

Nonlinear

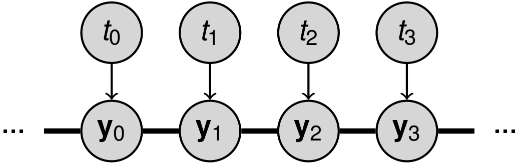
The world is nonlinear!

Linear dynamical systems are boring.

A Zoo of GP-Based Time Series Models

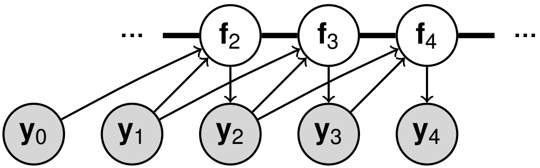
- Linear-Gaussian Auto-Regressive / State-Space
- Nonlinear Auto-Regressive Model with GP
- State-Space Model with Transition GP
- State-Space Model with Transition and Emission GPs
- GP-LVM with Correlated Latent Variables

0. Linear-Gaussian Auto-Regressive / State-Space



$$\mathbf{y}(t) \sim \mathcal{GP}\big(m(t), k(t, t')\big).$$

1. Nonlinear Auto-Regressive Model with GP

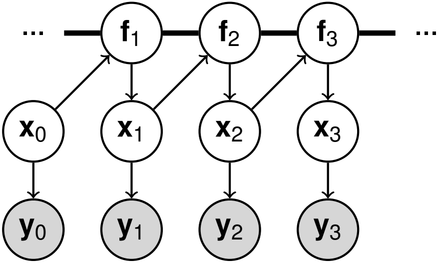


$$
\begin{aligned}
f(\mathbf{Y}) &\sim \mathcal{GP}\big(m_f(\mathbf{Y}), k_f(\mathbf{Y}, \mathbf{Y}')\big), \\
\mathbf{f}_t &= f(\mathbf{Y}_{t-1}), \\
\mathbf{y}_t \mid \mathbf{f}_t &\sim p(\mathbf{y}_t \mid \mathbf{f}_t, \theta),
\end{aligned}
$$

where

$$
\mathbf{Y}_{t-1} = \{\mathbf{y}_{t-1}, ..., \mathbf{y}_{t-n_y}\}.
$$

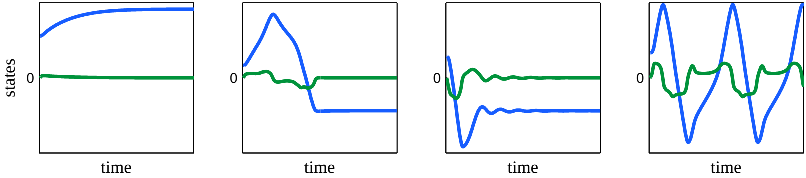
2. State-Space Model with Transition GP (GP-SSM)



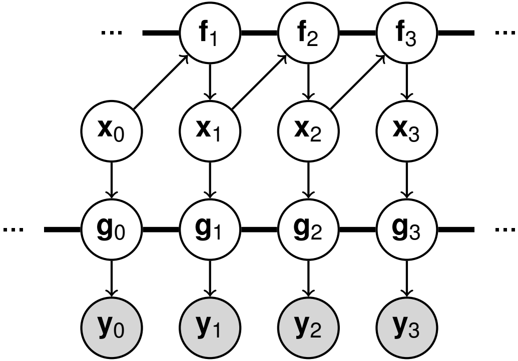
$$
\begin{aligned}
f(\mathbf{x}) &\sim \mathcal{GP}\big(m_f(\mathbf{x}), k_f(\mathbf{x}, \mathbf{x}')\big), \\
\mathbf{x}_0 &\sim p(\mathbf{x}_0), \\
\mathbf{f}_t &= f(\mathbf{x}_{t-1}), \\
\mathbf{x}_t \mid \mathbf{f}_t &\sim \mathcal{N}(\mathbf{x}_t \mid \mathbf{f}_t, \mathbf{Q}), \\
\mathbf{y}_t \mid \mathbf{x}_t &\sim p(\mathbf{y}_t \mid \mathbf{x}_t, \theta_y).
\end{aligned}
$$

2. State-Space Model with Transition GP (GP-SSM)

4 independent state trajectories from a 2-state GP-SSM with *fixed* hyperparameters.

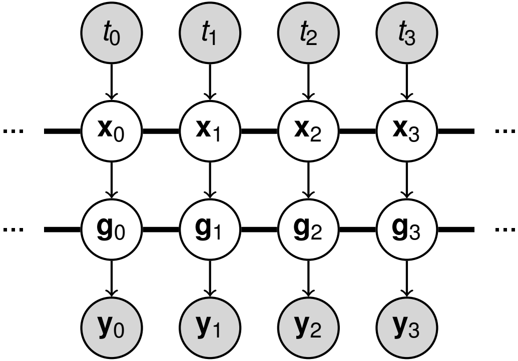


3. State-Space Model with Transition and Emission GPs (GP-SSM)



$$
\begin{aligned}
f(\mathbf{x}) &\sim \mathcal{GP}\big(m_f(\mathbf{x}), k_f(\mathbf{x}, \mathbf{x}')\big), & g(\mathbf{x}) &\sim \mathcal{GP}\big(m_g(\mathbf{x}), k_g(\mathbf{x}, \mathbf{x}')\big), \\
\mathbf{x}_0 &\sim p(\mathbf{x}_0), & \mathbf{g}_t &= g(\mathbf{x}_t), \\
\mathbf{f}_t &= f(\mathbf{x}_{t-1}), & \mathbf{y}_t \mid \mathbf{g}_t &\sim \mathcal{N}(\mathbf{y}_t \mid \mathbf{g}_t, \mathbf{R}). \\
\mathbf{x}_t \mid \mathbf{f}_t &\sim \mathcal{N}(\mathbf{x}_t \mid \mathbf{f}_t, \mathbf{Q}),
\end{aligned}
$$

4. GP-LVM with Correlated Latent Variables



$$
\begin{aligned}
x(t) &\sim \mathcal{GP}\big(m_x(t), k_x(t, t')\big), \\
g(\mathbf{x}) &\sim \mathcal{GP}\big(m_g(\mathbf{x}), k_g(\mathbf{x}, \mathbf{x}')\big), \\
\mathbf{x}_t &= x(t), \\
\mathbf{g}_t &= g(\mathbf{x}_t), \\
\mathbf{y}_t \mid \mathbf{g}_t &\sim \mathcal{N}\big(\mathbf{y}_t \mid \mathbf{g}_t, \beta^{-1}\mathbf{I}\big).
\end{aligned}
$$